Proceedings of the 17th IEEE Workshop on Signal and Power Integrity, SPI 2013, May 2013, pp. 151-154.

Per-Unit-Length Inductance Matrix Computations Using Modified Partial Inductances

Frédéric Broydé, Evelyne Clavelier
Excem
Maule, France
fredbroyde@eurexcem.com

Ronald De Smedt
Alcatel-Lucent Bell
Anvers, Belgium
ronald.de_smedt@alcatel-lucent.com

Lucie Broydé
ESIGELEC
Saint-Étienne-du-Rouvray, France
lbroyde@excem.fr

***Abstract* — We summarize the definition and the known properties of the modified partial inductances of parallel conductors. We show how modified partial inductances can be used to compute the dc per-unit-length (p.u.l.) inductance matrix of a multiconductor stripline interconnection. We then provide accurate asymptotic expansions for the dc p.u.l. inductance matrix of multiconductor microstrip or stripline structures, as the ground plane's breadth becomes large.**

I. Introduction

This paper is about the dc per-unit-length (p.u.l.) inductance matrix, denoted by $\mathbf{L}'_{DC}$, of a multiconductor interconnection having a specified cross-section comprising n transmission conductors (TCs) and a reference conductor (GC), such as the structures shown in Fig. 1 and Fig. 2.

In a recent paper, the modified partial inductances of parallel conductors have been introduced [1]. Exact analytical expressions for computing modified partial inductances were provided for the case where the cross-section of each conductor is a rectangle having an horizontal side. These formulas were used to directly compute $\mathbf{L}'_{DC}$ exactly, in the case where the cross-section of each TC and of the GC is a rectangle having an horizontal side. We note that this case covers the multiconductor microstrip structure of Fig. 1, but not the multiconductor stripline structure of Fig. 2. The paper [1] also investigated, for a simple multiconductor microstrip, the influence of the breadth of the ground plane, denoted by b. It was shown that all entries of $\mathbf{L}'_{DC}$ become large and are equivalent to $(\mu_0/(2\pi)) \ln b$ as $b \to \infty$.

Section II summarizes the definition and the known properties of modified partial inductances. Section III shows how they can be used to compute $\mathbf{L}'_{DC}$ in the case of a structure having a GC made of a single rectangular conductor. The new results of this paper are presented in the following sections. Section IV uses modified partial inductances to compute $\mathbf{L}'_{DC}$ in the case of a structure having a GC made of a two rectangular conductor, such as a multiconductor stripline. Section V provides asymptotic expansions of $\mathbf{L}'_{DC}$ as $b \to \infty$, accurate to $O(1/b^3)$, in the case of multiconductor microstrip and stripline interconnections.

II. Modified Partial Inductances

In this paper, we assume everywhere a permeability equal to the permeability of vacuum, denoted by μ_0. This Section is a summary of § II to § IV of [1], in which detailed proofs and adequate references can be found.

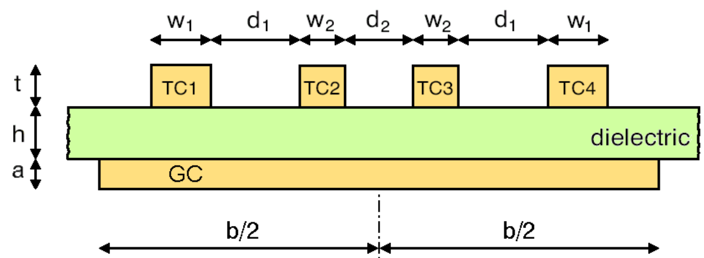

Fig. 1. Cross-section of a multiconductor microstrip interconnection comprising $n = 4$ transmission conductors (TCs), identified as TC1 to TC4, and a reference or ground conductor (GC).

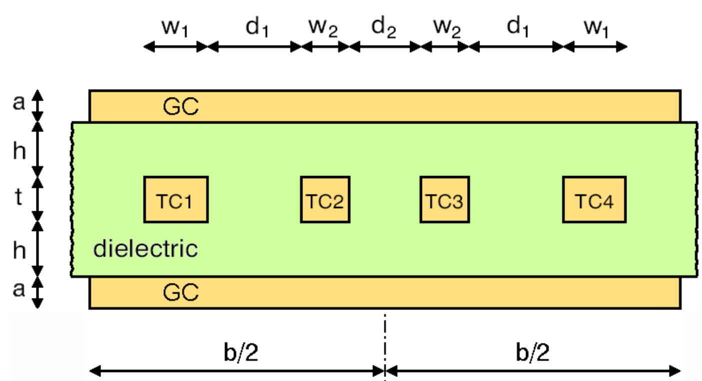

Fig. 2. Cross-section of a multiconductor stripline interconnection.

We consider the dc inductance matrix, denoted by $\mathbf{L}_{DC}$, of a circuit comprising n independent loops, made of straight segments which may be regarded as branches of the circuit. Each of said loop comprises: one long branch extending from $z = 0$ to $z = \mathcal{L}$ for the TC; another long branch extending from $z = 0$ to $z = \mathcal{L}$ for the GC; and two short branches, respectively at $z = 0$ and at $z = \mathcal{L}$. The traditional technique for the computation of $\mathbf{L}_{DC}$ uses the concept of partial inductance applied to the straight segments.

When we compute $\mathbf{L}_{DC}$, we find that, if $\mathcal{L}$ is much larger than the transverse dimensions of the loops, $\mathbf{L}_{DC}$ is nearly proportional to $\mathcal{L}$. Thus, $\mathbf{L}'_{DC}$ may be defined as

$$\mathbf{L}'_{DC} = \lim_{\mathcal{L} \to \infty} \frac{\mathbf{L}_{DC}}{\mathcal{L}} \tag{1}$$

because this limit exists and is nonzero.

Analytical expressions providing exact partial self-inductances and partial mutual inductances are available for the case where the cross-section of each conductor is a rectangle having an horizontal side [2]. They can be used to obtain exact formula for computing $\mathbf{L}_{DC}$. Unfortunately, these formula become numerically unstable when $\mathcal{L}$ is large. Consequently, it is not possible to directly use (1) to accurately compute $\mathbf{L}'_{DC}$.

To avoid this problem, the *modified partial inductances* of parallel conductors were defined. The modified partial inductance of the parallel conductors α and β extending from $z = 0$ to $z = \mathcal{L}$, denoted by $m'_{\alpha\beta}$, is

$$m'_{\alpha\beta} = \lim_{\mathcal{L}\to\infty}\left(\frac{m_{\alpha\beta}}{\mathcal{L}} - \frac{\mu_0}{2\pi}\ln\frac{2\mathcal{L}}{\mathcal{L}_0}\right) \tag{2}$$

where $\mathcal{L}_0$ is an arbitrary length and $m_{\alpha\beta}$ is the partial inductance between the conductors α and β. We note that this limit exists and is nonzero.

A loop α among the n independent loops involved in the definition of $\mathbf{L}'_{DC}$ contains two branches extending from $z = 0$ to $z = \mathcal{L}$, the branches of the subset N'_α. For a branch $p \in N'_\alpha$ let us define $\varepsilon_\alpha(p)$ by: $\varepsilon_\alpha(p) = 1$ if the branch p and the loop α have the same reference direction, $\varepsilon_\alpha(p) = -1$ otherwise. An entry $L'_{DC\,\alpha\beta}$ of $\mathbf{L}'_{DC}$ is given by

$$L'_{DC\alpha\beta} = \sum_{p\in N'_\alpha}\sum_{q\in N'_\beta}\varepsilon_\alpha(p)\varepsilon_\beta(q)\,m'_{pq} \tag{3}$$

Since (3) contains an equal number of terms with $\varepsilon_\alpha(p) = 1$ and with $\varepsilon_\alpha(p) = -1$, the result of (3) is independent of $\mathcal{L}_0$ used in (2).

Let $W_\alpha > 0$ be the width and $T_\alpha > 0$ be the thickness of the conductor α, which has a uniform rectangular cross-section. The modified partial self-inductance of this conductor is

$$m'_{\alpha\alpha} = \ell'(T_\alpha, W_\alpha) \tag{4}$$

where

$$\begin{aligned}&\ell'(x,y) = \ell'(y,x) = \\ &\frac{\mu_0}{4\pi}\left(-\ln\frac{x^2+y^2}{\mathcal{L}_0^2} - \frac{4}{3}\left\{\frac{x}{y}\tan^{-1}\frac{y}{x} + \frac{y}{x}\tan^{-1}\frac{x}{y}\right\}\right. \\ &\left. + \frac{1}{6}\left\{\frac{x^2}{y^2}\ln\left(1+\frac{y^2}{x^2}\right) + \frac{y^2}{x^2}\ln\left(1+\frac{x^2}{y^2}\right)\right\} + \frac{13}{6}\right)\end{aligned} \tag{5}$$

For a square cross-section of side S_α, we get

$$\begin{aligned} m'_{\alpha\alpha} &= \frac{\mu_0}{4\pi}\left(-\ln\frac{2S_\alpha^2}{\mathcal{L}_0^2} + \frac{\ln 2 - 2\pi}{3} + \frac{13}{6}\right) \\ &\approx \frac{\mu_0}{4\pi}\left(-\ln\frac{2S_\alpha^2}{\mathcal{L}_0^2} + 0.303321\right)\end{aligned} \tag{6}$$

which should be used in the place of the equation (23) of [1], which contains an incorrect constant.

The axis of the conductor α, along which the current flows, is

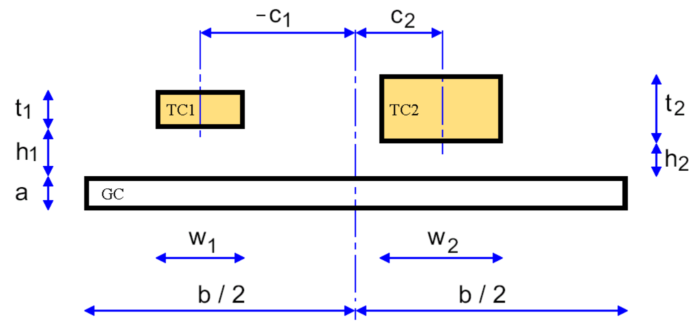

Fig 3. The generic multiconductor microstrip configuration considered in Section III.

parallel to the z axis. The reference direction is the direction of increasing z. The cross section of the conductor α extends from $x = x_\alpha$ to $x = x_\alpha + T_\alpha$ and from $y = y_\alpha$ to $y = y_\alpha + W_\alpha$. For this conductor, we define the two vectors

$$\mathbf{X}_\alpha = \begin{pmatrix} x_\alpha \\ x_\alpha + T_\alpha \end{pmatrix} \quad\text{and}\quad \mathbf{Y}_\alpha = \begin{pmatrix} y_\alpha \\ y_\alpha + W_\alpha \end{pmatrix} \tag{7}$$

The modified partial inductance between two such parallel conductors, the cross section of each of which is a rectangle having an horizontal side, is given by

$$m'_{\alpha\beta} = \frac{\displaystyle\sum_{I=1}^{2}\sum_{J=1}^{2}\sum_{L=1}^{2}\sum_{M=1}^{2}(-1)^{I+J+L+M}\left(X_{\alpha I} - X_{\beta L}\right)^2\left(Y_{\alpha J} - Y_{\beta M}\right)^2 m'_{I,J,L,M}}{4\,T_\alpha T_\beta W_\alpha W_\beta} \tag{8}$$

where

$$m'_{I,J,L,M} = \begin{cases} 0 \text{ if } \left(Y_{\alpha J} - Y_{\beta M}\right)\left(X_{\alpha I} - X_{\beta L}\right) = 0 \\ \ell'\left(\left|Y_{\alpha J} - Y_{\beta M}\right|, \left|X_{\alpha I} - X_{\beta L}\right|\right) \text{ else}\end{cases} \tag{9}$$

The modified partial mutual inductance formula (8) comprises 16 terms containing $\ell'(x, y)$ defined by (5). In the case $\alpha = \beta$, only 4 terms are nonzero and the nonzero terms are equal, so that (8) and (9) give the same result as (5).

III. Computation of DC P.U.L. Inductance Matrix — Case of a Single Ground Plane

Since there is no interaction between the dc current distribution in the conductors, the generic configuration shown in Fig. 3 can be used to compute the entries of any dc p.u.l. inductance matrix having a GC made of a single rectangular conductor. In Fig. 3, we note that h_1, t_1, w_1, h_2, t_2, c_2, w_2, a and b are positive and c_1 is negative. We have to determine (7) for 3 different conductors. We use, for $\alpha \in \{1, 2\}$,

$$\mathbf{X}_\alpha = \begin{pmatrix} h_\alpha \\ h_\alpha + t_\alpha \end{pmatrix}, \qquad \mathbf{Y}_\alpha = \begin{pmatrix} c_\alpha - \dfrac{w_\alpha}{2} \\ c_\alpha + \dfrac{w_\alpha}{2} \end{pmatrix} \tag{10}$$

and

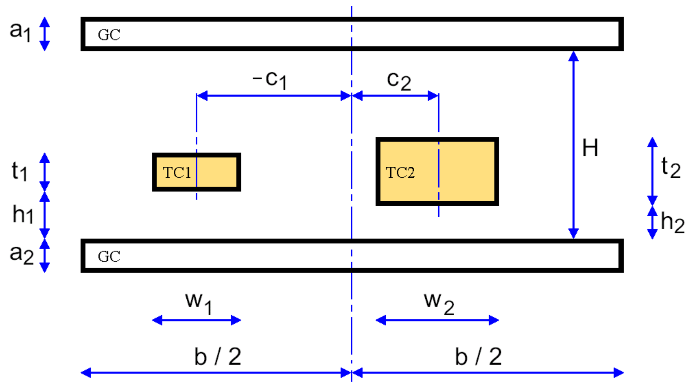

Fig. 4. The generic multiconductor stripline configuration considered in Section IV.

$$\mathbf{X}_3 = \begin{pmatrix} -a \\ 0 \end{pmatrix}, \qquad \mathbf{Y}_3 = \begin{pmatrix} -\dfrac{b}{2} \\ \dfrac{b}{2} \end{pmatrix} \tag{11}$$

For this problem, $\mathbf{L}'_{DC}$ is exactly given by

$$L'_{DC\alpha\alpha} = m'_{\alpha\alpha} + m'_{33} - 2m'_{\alpha 3} \tag{12}$$

and

$$L'_{DC12} = L'_{DC21} = m'_{12} - m'_{13} - m'_{23} + m'_{33} \tag{13}$$

The results of (12) and (13) are independent of $\mathcal{L}_0$ used in (5). These formula have for instance been applied to the multiconductor microstrip interconnection shown in Fig. 1, in the case $t = h = a = w_1 = w_2 = d_1 = d_2 = 50$ µm, to obtain the Fig. 3 of [3].

IV. Computation of DC P.U.L. Inductance Matrix — Case of Two Ground Planes

If the GC is made of two conductors of rectangular cross section, each of them may be regarded as a separate branch, of respective dc p.u.l. resistance $R'_{GC\,1}$ and $R'_{GC\,2}$. A current I injected in the GC splits into a current $I_{GC\,1}$ in the first of these conductors and a current $I_{GC\,2}$ in the second, $I_{GC\,1}$ and $I_{GC\,2}$ being given by

$$I_{GC1} = k_1 I \quad \text{with} \quad k_1 = \frac{R'_{GC2}}{R'_{GC1} + R'_{GC2}}$$
$$I_{GC2} = k_2 I \quad \text{with} \quad k_2 = \frac{R'_{GC1}}{R'_{GC1} + R'_{GC2}} \tag{14}$$

We can then follow the reasoning used in [2, § 4], to obtain $\mathbf{L}'_{DC}$. In the special case of two identical ground planes, we have $k_1 = k_2 = 0.5$.

The generic configuration shown in Fig. 4 can be used to compute the entries of any dc p.u.l. inductance matrix having a GC made of two superimposed and identical rectangular conductors. In Fig. 4, h_1, t_1, w_1, h_2, t_2, c_2, w_2, a_1, a_2, b and H are positive and c_1 is negative. We have to determine (7) for 4 different conductors. We use (10) for $\alpha \in \{1, 2\}$, and

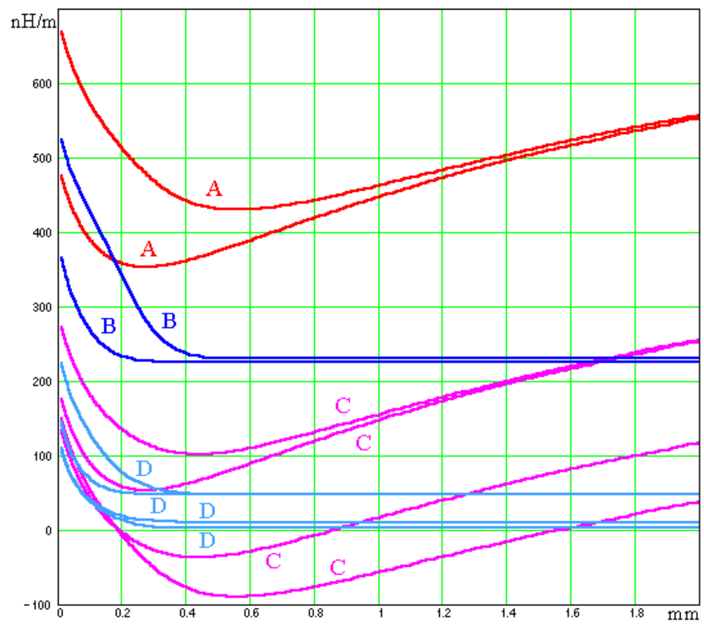

Fig. 5. For the multiconductor stripline interconnection, the dc p.u.l. self-inductances (2 curves A), the diagonal entries of $\mathbf{L}_0$ (2 curves B), the dc p.u.l. mutual inductances (4 curves C) and the non-diagonal entries of $\mathbf{L}_0$ (4 curves D) as a function of *b*.

$$\mathbf{X}_3 = \begin{pmatrix} -a_2 \\ 0 \end{pmatrix}, \mathbf{Y}_3 = \begin{pmatrix} -\dfrac{b}{2} \\ \dfrac{b}{2} \end{pmatrix}, \mathbf{X}_4 = \begin{pmatrix} H \\ H + a_1 \end{pmatrix}, \mathbf{Y}_4 = \begin{pmatrix} -\dfrac{b}{2} \\ \dfrac{b}{2} \end{pmatrix} \tag{15}$$

For this problem, $\mathbf{L}'_{DC}$ is exactly given by

$$L'_{DC\alpha\alpha} = m'_{\alpha\alpha} - m'_{\alpha 3} - m'_{\alpha 4} + \frac{m'_{33} + m'_{44} + 2m'_{34}}{4} \tag{16}$$

and

$$L'_{DC12} = m'_{12} - \frac{m'_{13} + m'_{23} + m'_{14} + m'_{24}}{2} + \frac{m'_{33} + m'_{44} + 2m'_{34}}{4} \tag{17}$$

The results of (16) and (17) are independent of $\mathcal{L}_0$ used in (5). The Fig. 5 shows the entries of $\mathbf{L}'_{DC}$ and of the high-frequency p.u.l. external inductance matrix, denoted by $\mathbf{L}'_0$, computed as a function of b, for the multiconductor stripline interconnection of Fig. 2, in the case $t = h = a = w_1 = w_2 = d_1 = d_2 = 50$ µm. The entries of $\mathbf{L}'_{DC}$ were obtained using (10) and (15) to (17). The entries of $\mathbf{L}'_0$ were computed by the method of moment using pulse expansion and 1400 matching points. Among the differences between $\mathbf{L}'_{DC}$ and $\mathbf{L}'_0$, we note that: negative nondiagonal entries exist in $\mathbf{L}'_{DC}$ but not in $\mathbf{L}'_0$; and, for $b > 2(w_1 + d_1 + w_2) + d_2 + 2h = 450$ µm, $\mathbf{L}'_0$ is close to a limit obtained for an infinite ground plane, whereas such a limit does not exist for $\mathbf{L}'_{DC}$.

V. Asymptotic Expansions for a Broad Ground Plane

We now want to explore the behavior of $\mathbf{L}'_{DC}$ as $b \to \infty$, using accurate asymptotic expansions. We omit the corresponding derivations. For the generic multiconductor microstrip configuration of Fig. 3, $\alpha \in \{1, 2\}$ and $\beta \in \{1, 2\}$, we obtain:

$$L'_{DC\alpha\beta} = L'_{DC\beta\alpha} = m'_{\alpha\beta} + \frac{\mu_0}{4\pi}\left[2\ln\frac{b}{4\mathcal{L}_0} + 1 + \frac{E_{\alpha\beta}}{b} + \frac{a^2}{3b^2}\ln\frac{b}{a} + \frac{F_{\alpha\beta}}{b^2}\right] + O\left(\frac{1}{b^3}\right) \tag{18}$$

where

$$\begin{cases} E_{\alpha\beta} = \pi\,\dfrac{4a + 3t_\alpha + 6h_\alpha + 3t_\beta + 6h_\beta}{3} \\ F_{\alpha\beta} = -\dfrac{71a^2}{36} \\ \quad -\dfrac{12h_\alpha^2 + 12(a+t_\alpha)h_\alpha + 6at_\alpha + 4t_\alpha^2 - w_\alpha^2 - 12c_\alpha^2}{3} \\ \quad -\dfrac{12h_\beta^2 + 12(a+t_\beta)h_\beta + 6at_\beta + 4t_\beta^2 - w_\beta^2 - 12c_\beta^2}{3} \end{cases} \tag{19}$$

For the generic multiconductor stripline configuration of Fig. 4, in the case $a_1 = a_2 = a$, for $\alpha \in \{1, 2\}$ and $\beta \in \{1, 2\}$, we get:

$$L'_{DC\alpha\beta} = m'_{\alpha\beta} + \frac{\mu_0}{4\pi}\left[\begin{array}{l} 2\ln\dfrac{b}{4\mathcal{L}_0} + 1 + \dfrac{U}{b} + \dfrac{a^2}{6b^2}\ln\dfrac{b}{a} \\ +\dfrac{(H+2a)^4}{12a^2b^2}\ln\dfrac{b}{H+2a} + \dfrac{H^4}{12a^2b^2}\ln\dfrac{b}{H} \\ -\dfrac{(H+a)^4}{6a^2b^2}\ln\dfrac{b}{H+a} + \dfrac{V_{\alpha\beta}}{b^2} \end{array}\right] + O\left(\frac{1}{b^3}\right) \tag{20}$$

where

$$\begin{cases} U = \pi\,\dfrac{2a + 3H}{3} \\ V_{\alpha\beta} = \dfrac{75H^2 + 150aH + 4a^2}{36} \\ \quad -\dfrac{1}{3}\begin{pmatrix} 6(H - h_\alpha - t_\alpha)^2 + 6h_\alpha^2 + 6(a+t_\alpha)(H - t_\alpha) \\ +6at_\alpha + 4t_\alpha^2 - w_\alpha^2 - 12c_\alpha^2 \end{pmatrix} \\ \quad -\dfrac{1}{3}\begin{pmatrix} 6(H - h_\beta - t_\beta)^2 + 6h_\beta^2 + 6(a+t_\beta)(H - t_\beta) \\ +6at_\beta + 4t_\beta^2 - w_\beta^2 - 12c_\beta^2 \end{pmatrix} \end{cases} \tag{21}$$

According to (18) and (20), all entries of $\mathbf{L}'_{DC}$ become large and are equivalent to $(\mu_0/(2\pi)) \ln b$ as $b \to \infty$. This corresponds to an oblique asymptote in a semi-log plot, presenting a slope of about 461nH per decade of b. The curves A and C of Fig. 6 show the entries of $\mathbf{L}'_{DC}$, computed as a function of b, for the multiconductor stripline interconnection considered in Section IV. The curves B and D of Fig. 6 show the approximate values given by (20) and (21). The agreement is good for $b > 0.6$ mm.

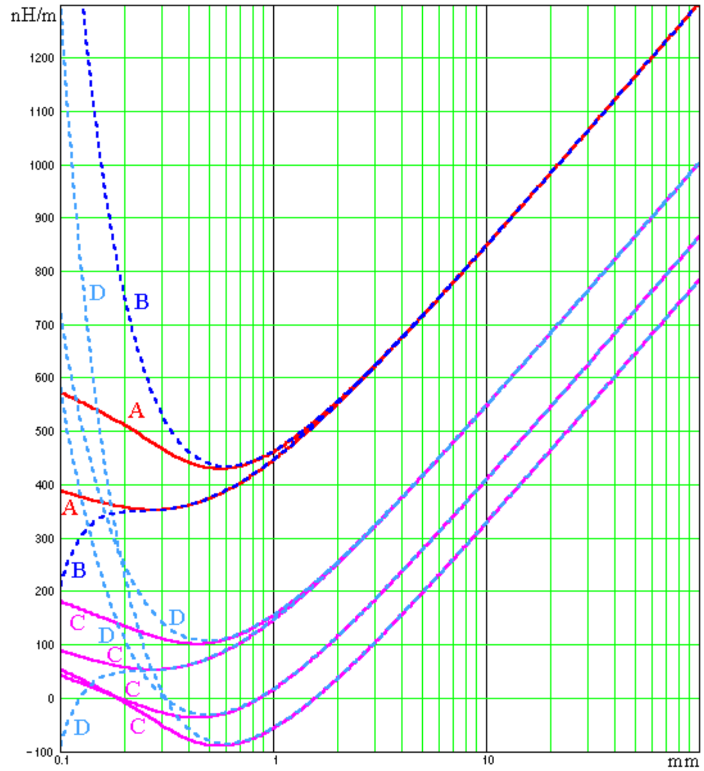

Fig. 6. For the multiconductor stripline interconnection, the dc p.u.l. self-inductances (2 curves A), their asymptotic expansions (2 curves B), the dc p.u.l. mutual inductances (4 curves C) and their asymptotic expansions (4 curves D) as a function of *b*.

VI. CONCLUSION

Modified partial inductances can be used to compute $\mathbf{L}'_{DC}$ of a muticonductor microstrip or stripline interconnection. We have computed accurate asymptotic expansions for large values of b.

We observe that, unlike $\mathbf{L}'_0$, $\mathbf{L}'_{DC}$ is only defined for a finite b, because all entries of $\mathbf{L}'_{DC}$ are equivalent to $(\mu_0/(2\pi)) \ln b$ as $b \to \infty$. This is a consequence of the dc current distribution in the GC, which is such that the average distance between the current lines in a TC and in the GC increases when b increases. Also, unlike $\mathbf{L}'_0$, $\mathbf{L}'_{DC}$ may have negative non-diagonal entries, in a range of values of b.

The results of this paper will be used to derive an approximate analytic model for the low-frequency p.u.l. inductance matrix, intended to allow a discussion of the underlying physics.

REFERENCES

[1] F. Broydé, E. Clavelier, L. Broydé, "A Direct Current Per-Unit-Length Inductance Matrix Computation Using Modified Partial Inductances", *Proc. of the CEM 2012 Int. Symp. on Electromagnetic Compatibility*, Rouen, France, April 2012, session 2C. Available: http://www.eurexcem.com.

[2] C. Hoer, C. Love, "Exact Inductance Equations for Rectangular Conductors With Applications to More Complicated Geometries", *J. Res. Nat. Bur. Stand.*, Vol. 69C, No. 2, pp. 127-137, April-June 1965.

[3] F. Broydé, E. Clavelier, D. De Zutter, D. Vande Ginste, "A discussion of an Analytical Per-Unit-Length Impedance Matrix Model", *Proc. IEEE 21st Conference on Electrical Performance of Electronic Packaging and Systems, EPEPS 2012*, Tempe, pp. 113-116, Oct. 21-24, 2012.