Proc. of the CEM 2012 Int. Symp. on Electromagnetic Compatibility,
Rouen, 25-27 April 2012.

A Direct Current Per-Unit-Length Inductance Matrix Computation Using Modified Partial Inductances

Frédéric Broydé, Evelyne Clavelier
Tekcem
Maule, France
e-mail: fredbroyde@tekcem.com, eclavelier@tekcem.com

Lucie Broydé
ESIGELEC
Saint-Étienne-du-Rouvray, France
email: lbroyde@excem.fr

***Abstract* — We introduce the new concept of the modified partial inductances of parallel conductors. For conductors of rectangular cross-section, we obtain exact formulas for computing the modified partial self-inductance and the modified partial mutual inductance. These results can be used to obtain the dc per-unit-length self- and mutual inductances which are needed in transmission line and multiconductor transmission line models of lossy interconnections.**

I. INTRODUCTION

In simulations for EMC or signal integrity assessment, it is often adequate to use a uniform transmission line (TL) or a uniform multiconductor transmission line (MTL) model for an interconnection [1]. A parameter of the TL model is the per-unit-length (p.u.l.) impedance of the 2-conductor interconnection, denoted by Z', while the corresponding parameter of the MTL model is the p.u.l. impedance matrix of the multiconductor interconnection, denoted by $\mathbf{Z}'$.

The dc p.u.l. inductance of the TL, denoted by L'_{DC}, and the dc p.u.l. inductance matrix of the MTL, denoted by $\mathbf{L}'_{DC}$, are computed for the dc current distribution in the conductors of the interconnection. This current distribution is homogeneous in the case of straight homogeneous conductors.

The current distribution in the conductors may be considered as frequency independent at any frequency f smaller than an onset frequency f_o at which the skin effect, the edge effect (crowding of currents at the edges of a conductor) and/or the proximity effect (modification of the current distribution caused by nearby conductors) start to significantly modify Z' or $\mathbf{Z}'$. Thus, for $f < f_o$ we have $Z' \approx R'_{DC} + 2\pi f L'_{DC}$ or $\mathbf{Z}' \approx \mathbf{R}'_{DC} + 2\pi f \mathbf{L}'_{DC}$, where R'_{DC} is the dc p.u.l. resistance of the TL and $\mathbf{R}'_{DC}$ is the dc p.u.l. resistance matrix of the MTL. Consequently, when losses are taken into account in a TL or MTL model, L'_{DC} or $\mathbf{L}'_{DC}$ is an important parameter of the interconnection.

This paper is about a new approach for the computation of the dc p.u.l. self-inductances, i.e. L'_{DC} or the diagonal entries of $\mathbf{L}'_{DC}$, and for the computation of dc p.u.l. mutual inductances, i.e. the nondiagonal entries of $\mathbf{L}'_{DC}$. We will obtain exact formulas for the case of conductors of rectangular cross-section which are typical of configurations commonly found in printed circuit boards and multi-chip modules, such as the ones shown in Fig. 1 and Fig. 2.

The traditional technique for the computation of dc self- and mutual inductances involving loops made of straight segments uses the concept of partial inductance applied to the straight segments [2] [3] [4]. Unfortunately, the p.u.l. partial inductances

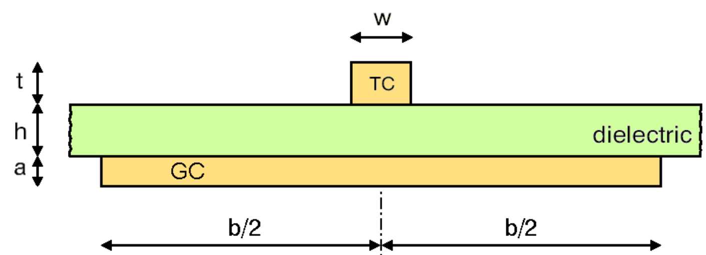

Fig. 1. Cross-section of a microstrip interconnection. TC is the transmission conductor and GC is the reference conductor.

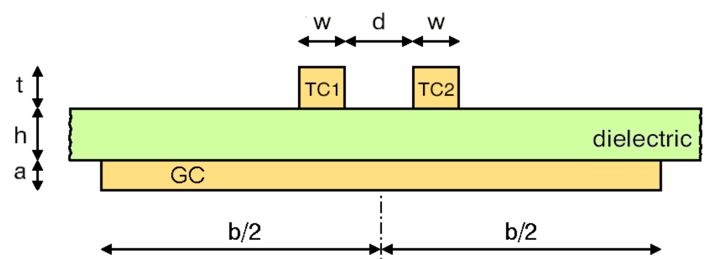

Fig. 2. Cross-section of a multiconductor microstrip interconnection. TC1 and TC2 are the transmission conductors.

of parallel segments of infinite length cannot be defined, so that partial inductance cannot be used to directly obtain L'_{DC} or $\mathbf{L}'_{DC}$. This question is covered in Section II. We then introduce the new concept of the modified partial inductances of parallel conductors, in Section III. For conductors of rectangular cross-section, exact formulas for the modified partial self-inductance and the modified partial mutual inductance are derived in Section IV. In Section V, these results are used to obtain L'_{DC} for the geometry of Fig. 1 and $\mathbf{L}'_{DC}$ for the geometry of Fig. 2. The case of a broad ground plane, i.e. $b \to \infty$, is discussed in Section VI because this assumption is often used to derive the high-frequency parameters of the TL or MTL using the method of images.

II. PARTIAL INDUCTANCE REVISITED

In this paper, we assume everywhere a permeability equal to the permeability of vacuum, denoted by μ_0. Let us consider a circuit comprising n independent loops. Here, two loops may have a part in common. We want to compute the inductance matrix between the loops. Assuming the conservation of current in each loop, we use I_α to denote the current in the loop α, in which a reference direction has been defined. We use $\mathbf{B}_{L\alpha}$ to denote the magnetic induction produced anywhere in space by I_α. In the quasi-static approximation (in a narrow sense), the magnetic energy of the circuit is, in the time domain, given by [5, § 5.12] [6, § 8.02]

$$U_m = \frac{1}{2\mu_0} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \iiint_V \mathbf{B}_{L\alpha} \cdot \mathbf{B}_{L\beta}\ dv \tag{1}$$

where the volume integral must be evaluated over all space, denoted by V. The inductances are defined by

$$L_{\alpha\beta}\, I_\alpha\, I_\beta = \frac{1}{\mu} \iiint_V \mathbf{B}_{L\alpha} \cdot \mathbf{B}_{L\beta}\ dv \tag{2}$$

In the case $\alpha = \beta$, $L_{\alpha\alpha}$ is the self-inductance of the loop α. If $\alpha \neq \beta$, $L_{\alpha\beta} = L_{\beta\alpha}$ is the mutual inductance between the loops α and β. The inductance matrix between loops is $\mathbf{L} = [L_{\alpha\beta}]$. We have

$$U_m = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} L_{\alpha\beta}\, I_\alpha\, I_\beta \tag{3}$$

It may be shown that $L_{\alpha\beta}$ is given by

$$L_{\alpha\beta} = \frac{\mu_0}{4\pi\, I_\alpha I_\beta} \iiint_{V_{L\alpha}} \iiint_{V_{L\beta}} \frac{\mathbf{j}_{L\alpha} \cdot \mathbf{j}_{L\beta}}{r}\ dv_{L\beta}\ dv_{L\alpha} \tag{4}$$

where $\mathbf{j}_{L\alpha}$ denotes the current density associated to I_α, $V_{L\alpha}$ denotes the volume of the loop α, $dv_{L\alpha}$ denotes a volume element of the loop α and r is the distance between the volume elements.

We now consider that each loop comprises one or more branches. Some branches may belong to two or more loops. Let us number from 1 to N the branches forming the loops and let us use i_α to denote the current in the branch α, in which a reference direction has been defined. Let us use $\mathbf{B}_{b\alpha}$ to denote the magnetic induction produced anywhere in space by i_α. In the quasi-static approximation, the magnetic energy of the circuit is given by

$$U_m = \frac{1}{2\mu_0} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \iiint_V \mathbf{B}_{b\alpha} \cdot \mathbf{B}_{b\beta}\ dv \tag{5}$$

where the volume integral must be evaluated over all space, as in (1). The partial inductances are defined by

$$m_{\alpha\beta}\, i_\alpha\, i_\beta = \frac{1}{\mu_0} \iiint_V \mathbf{B}_{b\alpha} \cdot \mathbf{B}_{b\beta}\ dv \tag{6}$$

In the case $\alpha = \beta$, $m_{\alpha\alpha}$ is the partial self-inductance of the branch α. If $\alpha \neq \beta$, $m_{\alpha\beta} = m_{\beta\alpha}$ is the partial mutual inductance between the branches α and β. Thus, we have

$$U_m = \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} m_{\alpha\beta}\, i_\alpha\, i_\beta \tag{7}$$

It may be shown that $m_{\alpha\beta}$ is given by

$$m_{\alpha\beta} = \frac{\mu_0}{4\pi\, i_\alpha i_\beta} \iiint_{V_{b\alpha}} \iiint_{V_{b\beta}} \frac{\mathbf{j}_{b\alpha} \cdot \mathbf{j}_{b\beta}}{r}\ dv_{b\beta}\ dv_{b\alpha} \tag{8}$$

where $\mathbf{j}_{b\alpha}$ denotes the current density associated to i_α, $V_{b\alpha}$ denotes the volume of the branch α and $dv_{b\alpha}$ denotes a volume element of the branch α. We observe that $\mathbf{j}_{b\alpha}$ is also the total current density in the branch α.

A loop α is formed by the branches of the subset $N_\alpha \subset \{1,..., N\}$. For a branch $p \in N_\alpha$, let us define $\varepsilon_\alpha(p)$ by: $\varepsilon_\alpha(p) = 1$ if the branch p and the loop α have the same reference direction, $\varepsilon_\alpha(p) = -1$ otherwise. Using (3) and (7) we can prove that a simple relation exists between the mutual and self-inductances of the loops and the partial mutual and self-inductances of the branches:

$$L_{\alpha\beta} = \sum_{p \in N_\alpha} \sum_{q \in N_\beta} \varepsilon_\alpha(p)\varepsilon_\beta(q)\, m_{pq} \tag{9}$$

In the case where the branch α has a uniform cross section and where an homogeneous and uniform current density exists in the branch α, let us use $a_{b\alpha}$ to denote the area of this cross section. Here, $m_{\alpha\beta}$ is given by [7, eq. (2) and (3)] [8, eq. (12)] [9, eq. (2)]:

$$m_{\alpha\beta} = \frac{\displaystyle\iint_{C_{b\alpha}} \iint_{C_{b\beta}} \frac{\mu_0}{4\pi} \int_{P_{b\alpha}} \int_{P_{b\beta}} \frac{d\,\mathbf{l}_{b\beta} \cdot d\,\mathbf{l}_{b\alpha}}{r}\, dS_{b\beta}\, dS_{b\alpha}}{a_{b\alpha} a_{b\beta}} \tag{10}$$

where $P_{b\alpha}$ denotes the path of the branch α, $C_{b\alpha}$ denotes the cross-section of the branch α, $\mathbf{l}_{b\alpha}$ denotes the vector displacement along the path of the branch α, and $S_{b\alpha}$ denotes the surface element of the branch α. In (10), the integrand of the surface integrals is the partial mutual inductance between two filaments (i.e., conductors of infinitesimal cross-section) given by the Neumann formula, and this integrand is averaged over the cross-section of the two branches. We note that, in the case where r may become arbitrarily small (e.g., when $\alpha = \beta$), the singularity in the integrand does not cause a problem for computing (10), as long as both branches are of finite length. We also note that the result (10) becomes the definition of Paul for partial mutual inductances [10] [11], in the special case of filaments.

In the special case where the branches are straight conductors of uniform cross-section, parallel to the z axis and extending from $z = 0$ to $z = \mathcal{L}$, the partial mutual inductance between the filaments is given by [2, § 4] [3, § 8] [4, ch. 5]:

$$\frac{\mu_0}{4\pi} \int_{P_{b\alpha}} \int_{P_{b\beta}} \frac{d\,\mathbf{l}_{b\beta} \cdot d\,\mathbf{l}_{b\alpha}}{r} = \frac{\mu_0}{4\pi} \int_0^{\mathcal{L}} \int_0^{\mathcal{L}} \frac{d\,z_\beta\ dz_\alpha}{r} = \frac{\mu_0 \mathcal{L}}{2\pi} \left[\ln\left(\frac{\mathcal{L}}{d} + \sqrt{1 + \left(\frac{\mathcal{L}}{d}\right)^2} \right) - \sqrt{1 + \left(\frac{d}{\mathcal{L}}\right)^2} + \frac{d}{\mathcal{L}} \right] \tag{11}$$

where d is the distance between the filaments. In the case where $\mathcal{L} >> d$, we note that

$$\frac{\mu_0}{4\pi} \int_0^{\mathcal{L}} \int_0^{\mathcal{L}} \frac{d\,z_\beta\ dz_\alpha}{r} \sim \frac{\mu_0 \mathcal{L}}{2\pi} \ln\left(\frac{2\mathcal{L}}{d}\right) \tag{12}$$

where $\sim$ means “asymptotically equivalent to”. Thus,

$$m_{\alpha\beta} \sim \frac{\mu_0 \mathcal{L}}{2\pi} \ln\left(\frac{2\mathcal{L}}{d}\right) \tag{13}$$

It is consequently not possible to define a p.u.l. partial self-inductance or a p.u.l. partial mutual inductance, as explained in the introduction.

When we compute an inductance matrix **L** between loops each containing two of said branches extending from $z = 0$ to $z = \mathcal{L}$, we find that (9) contains an equal number of terms with $\varepsilon_\alpha(p) = 1$ and with $\varepsilon_\alpha(p) = -1$ applying to the partial mutual and self-inductances of said branches. We then find that, if $\mathcal{L}$ is much larger than the transverse dimensions of the loops, **L** is nearly proportional to $\mathcal{L}$, because the terms in the form of (13) cancel out in (9). Thus, the p.u.l. inductance matrix, denoted by $\mathbf{L}' = [L'_{\alpha\beta}]$, may be defined as

$$\mathbf{L}' = \lim_{\mathcal{L}\to\infty} \frac{\mathbf{L}}{\mathcal{L}} \tag{14}$$

because this limit exists and is nonzero. To obtain (11) to (14), we only assumed straight parallel conductors of uniform cross-section and an homogeneous and uniform current density. Thus, all results of this Section II are applicable to the computation of $\mathbf{L}'_{DC}$.

III. Modified Partial Inductances

Formulas providing exact partial self-inductances and partial mutual inductances for parallel conductors of uniform rectangular cross-section are available [7] [8] [9] [12]. Unfortunately, they become numerically unstable when the length of the interconnection, denoted by $\mathcal{L}$, is large [12]. Consequently, it is not possible to directly use (9) and (14) to accurately compute $\mathbf{L}'_{DC}$.

To avoid this problem, we define the *modified partial inductances* of the parallel conductors α and β, denoted by $m'_{\alpha\beta}$, in the following way

$$m'_{\alpha\beta} = \lim_{\mathcal{L}\to\infty}\left(\frac{m_{\alpha\beta}}{\mathcal{L}} - \frac{\mu_0}{2\pi}\ln\frac{2\mathcal{L}}{\mathcal{L}_0}\right) \tag{15}$$

where $\mathcal{L}_0$ is an arbitrary length. We note that this limit exists and is nonzero. When we compute $\mathbf{L}'_{DC} = [L'_{DC\,\alpha\beta}]$ between loops each comprising two parallel conductors extending from $z = 0$ to $z = \mathcal{L}$ and regarded as a branch, we obtain

$$L'_{DC\alpha\beta} = \sum_{p\in N'_\alpha} \sum_{q\in N'_\beta} \varepsilon_\alpha(p)\varepsilon_\beta(q) m'_{pq} \tag{16}$$

where the loop α contains two branches extending from $z = 0$ to $z = \mathcal{L}$, the branches of the subset $N'_\alpha \subset \{1,\ldots, N\}$. To obtain (16), we have used (9), (14), (15), the fact that (9) contains an equal number of terms with $\varepsilon_\alpha(p) = 1$ and with $\varepsilon_\alpha(p) = -1$ applying to the partial mutual and self-inductances of said branches, and the fact that the other branches of the loop have a zero contribution to the loop inductance as $\mathcal{L} \to \infty$. Thus, an entry of $\mathbf{L}'_{DC}$ is easily obtained if we know the relevant modified partial inductances. Note that the result of (16) is independent of $\mathcal{L}_0$ used in (15).

A modified partial inductance may be determined as follows. Using (10) and (15), we get

$$m'_{\alpha\beta} = \frac{\iint_{C_{b\alpha}} \iint_{C_{b\beta}} \lim_{\mathcal{L}\to\infty}\left(\frac{m_{\alpha\beta}}{\mathcal{L}} - \frac{\mu_0}{2\pi}\ln\frac{2\mathcal{L}}{\mathcal{L}_0}\right) dS_{b\beta}\, dS_{b\alpha}}{a_{b\alpha} a_{b\beta}} \tag{17}$$

Using (11), we obtain

$$m'_{\alpha\beta} = \frac{-\mu_0 \iint_{C_{b\alpha}} \iint_{C_{b\beta}} \left(1 + \ln\frac{d}{\mathcal{L}_0}\right) dS_{b\beta}\; dS_{b\alpha}}{2\pi\, a_{b\alpha} a_{b\beta}} \tag{18}$$

which is also equal to

$$m'_{\alpha\beta} = \frac{-\mu_0}{2\pi}\left(1 + \frac{\iint_{C_{b\alpha}} \iint_{C_{b\beta}} \ln\frac{d}{\mathcal{L}_0}\; dS_{b\beta}\; dS_{b\alpha}}{a_{b\alpha} a_{b\beta}}\right) \tag{19}$$

The term comprising the 4-fold integral in (19) is the natural logarithm of the geometrical mean distance defined by Maxwell, normalized with $\mathcal{L}_0$ [3, § 9] [4, ch. 3] [13, § 691].

IV. Conductor of Rectangular Cross-Section

The modified partial self-inductance of a straight conductor of uniform rectangular cross-section may be computed using the formula given by Maxwell for the geometrical mean distance of a rectangle from itself [13, § 692]. Let W_α be the width and T_α be the thickness of the rectangular conductor α. The result of Maxwell is:

$$\begin{aligned}&\frac{1}{a_{b\alpha}a_{b\alpha}} \iint_{C_{b\alpha}} \iint_{C_{b\alpha}} \ln\frac{d}{\mathcal{L}_0}\; dS_{b\alpha}\; dS_{b\alpha} = \ln\frac{\sqrt{W_\alpha^2 + T_\alpha^2}}{\mathcal{L}_0} \\ &-\frac{1}{6}\left(\frac{T_\alpha^2}{W_\alpha^2}\ln\sqrt{1+\frac{W_\alpha^2}{T_\alpha^2}} + \frac{W_\alpha^2}{T_\alpha^2}\ln\sqrt{1+\frac{T_\alpha^2}{W_\alpha^2}}\right) \\ &+\frac{2}{3}\left(\frac{T_\alpha}{W_\alpha}\tan^{-1}\frac{W_\alpha}{T_\alpha} + \frac{W_\alpha}{T_\alpha}\tan^{-1}\frac{T_\alpha}{W_\alpha}\right) - \frac{25}{12}\end{aligned} \tag{20}$$

If we insert (20) in (15), we obtain the exact formula

$$m'_{\alpha\alpha} = \ell'(T_\alpha, W_\alpha) \tag{21}$$

where

$$\begin{aligned}&\ell'(x,y) = \ell'(y,x) = \\ &\frac{\mu_0}{4\pi}\left(-\ln\frac{x^2+y^2}{\mathcal{L}_0^2} - \frac{4}{3}\left\{\frac{x}{y}\tan^{-1}\frac{y}{x} + \frac{y}{x}\tan^{-1}\frac{x}{y}\right\}\right. \\ &\left.+\frac{1}{6}\left\{\frac{x^2}{y^2}\ln\left(1+\frac{y^2}{x^2}\right) + \frac{y^2}{x^2}\ln\left(1+\frac{x^2}{y^2}\right)\right\} + \frac{13}{6}\right)\end{aligned} \tag{22}$$

This exact result may also be derived from the exact partial self-inductance formula of Ruehli [8, § 5] in the form given by Wu, Kuo and Chang [9]. For a square cross-section of side S_α , we get

$$
\begin{aligned}
m'_{\alpha\alpha} &= \frac{\mu_0}{4\pi}\left(-\ln\frac{2S_\alpha^{\;2}}{\mathcal{L}_0^{\;2}}+\frac{\ln 2-2\pi}{3}+\frac{13}{6}\right) \\
&\approx \frac{\mu_0}{4\pi}\left(-\ln\frac{2S_\alpha^{\;2}}{\mathcal{L}_0^{\;2}}+0.172615\right)
\end{aligned}
\tag{23}
$$

The modified partial mutual inductance between two conductors of uniform rectangular cross-section having an horizontal side could in principle be computed using the exact formula for the geometric mean distance between symmetrically placed rectangles established by Gray [14, pp. 296-303] and revised by Rosa [15, § 3], or the more general exact partial mutual inductance formula of Hoer and Love [7, § 2.5]. However, the former does not cover all the cases which we need to address and the latter does not look very convenient for a direct implementation of (15). For this reason, we have used the exact partial mutual inductance formula of Zhong and Koh [12], which leads to a simple result.

Let $\mathcal{L}_\alpha > 0$ be the length, $W_\alpha > 0$ be the width and $T_\alpha > 0$ be the thickness of the conductor α of uniform rectangular cross-section. The axis of the conductor, along which the current flows, is parallel to the z axis and the reference direction is the direction of increasing z. The conductor α extends from $x = x_\alpha$ to $x = x_\alpha + T_\alpha$, from $y = y_\alpha$ to $y = y_\alpha + W_\alpha$ and from $z = z_\alpha$ to $z = z_\alpha + \mathcal{L}_\alpha$. For this conductor, we define the three vectors

$$
\mathbf{X}_\alpha = \begin{pmatrix} x_\alpha \\ x_\alpha + T_\alpha \end{pmatrix},\ \mathbf{Y}_\alpha = \begin{pmatrix} y_\alpha \\ y_\alpha + W_\alpha \end{pmatrix},\ \mathbf{Z}_\alpha = \begin{pmatrix} z_\alpha \\ z_\alpha + \mathcal{L}_\alpha \end{pmatrix}
\tag{24}
$$

According to Zhong and Koh, the partial mutual inductance between the conductors α and β is given by

$$
m_{\alpha\beta} = \frac{\displaystyle\sum_{I=1}^{2}\sum_{J=1}^{2}\sum_{K=1}^{2}\sum_{L=1}^{2}\sum_{M=1}^{2}\sum_{N=1}^{2}(-1)^{I+J+K+L+M+N+1} f_{I,J,K,L,M,N}}{8\ T_\alpha\, T_\beta\, W_\alpha\, W_\beta}
\tag{25}
$$

where

$$
f_{I,J,K,L,M,N} = \begin{cases} 0 \ \text{ if } \ \left(Z_{\alpha K}-Z_{\beta N}\right)\left(Y_{\alpha J}-Y_{\beta M}\right)\left(X_{\alpha I}-X_{\beta L}\right)=0 \\ \left(X_{\alpha I}-X_{\beta L}\right)^2\left(Y_{\alpha J}-Y_{\beta M}\right)^2 \\ \quad \cdot\, \ell\left(\left|Z_{\alpha K}-Z_{\beta N}\right|, \left|Y_{\alpha J}-Y_{\beta M}\right|, \left|X_{\alpha I}-X_{\beta L}\right|\right) \text{ else} \end{cases}
\tag{26}
$$

where $\ell(z, y, x)$ denotes the partial self-inductance of a conductor of uniform rectangular cross-section of length z, width y and thickness x.

The case of interest for computing the modified partial inductances of parallel conductors involves two branches extending from $z = z_\alpha$ to $z = z_\alpha + \mathcal{L}_\alpha$ so that we must use $z_\alpha = z_\beta$ and $\mathcal{L} = \mathcal{L}_\alpha = \mathcal{L}_\beta$. In this case, we have

$$
m_{\alpha\beta} = \frac{\displaystyle\sum_{I=1}^{2}\sum_{J=1}^{2}\sum_{L=1}^{2}\sum_{M=1}^{2}(-1)^{I+J+L+M}\ g_{I,J,L,M}}{4\ T_\alpha\, T_\beta\, W_\alpha\, W_\beta}
\tag{27}
$$

where

$$
g_{I,J,L,M} = \begin{cases} 0 \quad \text{if} \quad \left(Y_{\alpha J}-Y_{\beta M}\right)\left(X_{\alpha I}-X_{\beta L}\right)=0 \\ \left(X_{\alpha I}-X_{\beta L}\right)^2\left(Y_{\alpha J}-Y_{\beta M}\right)^2 \\ \quad \cdot\, \ell\left(\mathcal{L}, \left|Y_{\alpha J}-Y_{\beta M}\right|, \left|X_{\alpha I}-X_{\beta L}\right|\right) \text{ else} \end{cases}
\tag{28}
$$

It can easily be shown that

$$
\frac{\displaystyle\sum_{I=1}^{2}\sum_{J=1}^{2}\sum_{L=1}^{2}\sum_{M=1}^{2}(-1)^{I+J+L+M}\left(X_{\alpha I}-X_{\beta L}\right)^2\left(Y_{\alpha J}-Y_{\beta M}\right)^2}{4\,T_\alpha\, T_\beta\, W_\alpha\, W_\beta} = 1
\tag{29}
$$

Using (15), (27) and (29), we obtain

$$
m'_{\alpha\beta} = \frac{\displaystyle\sum_{I=1}^{2}\sum_{J=1}^{2}\sum_{L=1}^{2}\sum_{M=1}^{2}(-1)^{I+J+L+M}\left(X_{\alpha I}-X_{\beta L}\right)^2\left(Y_{\alpha J}-Y_{\beta M}\right)^2 m'_{I,J,L,M}}{4\ T_\alpha\, T_\beta\, W_\alpha\, W_\beta}
\tag{30}
$$

where

$$
m'_{I,J,L,M} = \begin{cases} 0 \text{ if } \left(Y_{\alpha J}-Y_{\beta M}\right)\left(X_{\alpha I}-X_{\beta L}\right)=0 \\ \ell'\left(\left|Y_{\alpha J}-Y_{\beta M}\right|, \left|X_{\alpha I}-X_{\beta L}\right|\right) \text{ else} \end{cases}
\tag{31}
$$

The modified partial mutual inductance formula (30) comprises 16 terms containing $\ell'(x, y)$ defined by (22). In the case $\alpha = \beta$, only 4 terms are nonzero and the nonzero terms are equal, so that (30) and (31) give the same result as (21).

V. Computation of DC P.U.L. Inductances

In the case of a uniform interconnection having n transmission conductors (TCs) and a reference conductor (GC), the p.u.l. inductance matrix is of size $n \times n$. We number the TCs from 1 to n. The TC number $\alpha \in \{1,..., n\}$ corresponds to the branch α and forms the loop α with the GC. The GC corresponds to the branch $n + 1$. The p.u.l. inductance matrix is given by (16), (21), (30) and (31), the result being independent of the arbitrary length $\mathcal{L}_0$.

As a first example, we have computed L'_{DC} for the interconnection of Fig. 1, for which $n = 1$. By (16) we have

$$
L'_{DC} = m'_{11} + m'_{22} - 2m'_{12}
\tag{32}
$$

We can use

$$
\mathbf{X}_1 = \begin{pmatrix} h \\ h+t \end{pmatrix},\ \mathbf{Y}_1 = \begin{pmatrix} -w/2 \\ w/2 \end{pmatrix},\ \mathbf{X}_2 = \begin{pmatrix} -a \\ 0 \end{pmatrix},\ \mathbf{Y}_2 = \begin{pmatrix} -b/2 \\ b/2 \end{pmatrix}
\tag{33}
$$

The curve A of Fig. 3 shows L'_{DC} computed as a function of b, for $t = h = a = w = 50$ µm, using (32) and (33). The curve B of Fig. 3 shows the high-frequency (h.f.) p.u.l. external inductance for the same configuration, denoted by L'_0 and computed as explained in § V of [16], by the method of moment using pulse expansion and 620 matching points. We observe that L'_{DC} and L'_0 are completely different. In particular, as b increases above $w + 2(h + t) =$ 250 µm, L'_0 quickly approaches the limit obtained for an infinite ground plane, equal to 317.5 nH/m [16].

As a second example, we have computed $\mathbf{L}'_{DC}$ for the interconnection of Fig. 2, for which $n = 2$. By (16) we have, for $\alpha \in \{1, 2\}$

$$L'_{DC\alpha\alpha} = m'_{\alpha\alpha} + m'_{33} - 2m'_{\alpha 3} \tag{34}$$

and

$$L'_{DC12} = L'_{DC21} = m'_{12} - m'_{13} - m'_{23} + m'_{33} \tag{35}$$

We can use

$$\begin{cases} \mathbf{X}_1 = \begin{pmatrix} h \\ h+t \end{pmatrix}, & \mathbf{Y}_1 = \begin{pmatrix} -\dfrac{2w+d}{2} \\ -d/2 \end{pmatrix} \\ \mathbf{X}_2 = \begin{pmatrix} h \\ h+t \end{pmatrix}, & \mathbf{Y}_2 = \begin{pmatrix} d/2 \\ \dfrac{2w+d}{2} \end{pmatrix} \\ \mathbf{X}_3 = \begin{pmatrix} -a \\ 0 \end{pmatrix}, & \mathbf{Y}_3 = \begin{pmatrix} -b/2 \\ b/2 \end{pmatrix} \end{cases} \tag{36}$$

The curves A and C of Fig. 4 show the entries $L'_{DC\,11} = L'_{DC\,22}$ and $L'_{DC\,12} = L'_{DC\,21}$ of $\mathbf{L}'_{DC}$, respectively, computed as a function of b, for $t = h = a = w = d = 50$ µm, using (34) to (36). The curves B and D of Fig. 4 show the entries $L'_{0\,11} = L'_{0\,22}$ and $L'_{0\,12} = L'_{0\,21}$ of the h.f. p.u.l. external inductance matrix of the same configuration, computed by the method of moment using pulse expansion and 700 matching points. We see that $\mathbf{L}'_{DC}$ and $\mathbf{L}'_0$ are completely different. In particular, when b increases above $2w + d + 2(h + t) = 350$ µm, $\mathbf{L}'_0$ quickly approaches the limit obtained for an infinite ground plane.

VI. Asymptotic Expansions for a Broad Ground Plane

It can be shown that, for $y \to \infty$, we have

$$\ell'(x,y) = \frac{\mu_0}{4\pi}\left(2\ln\frac{L_0}{y} + 1 - \frac{2\pi}{3}\frac{x}{y} + \frac{x^2}{3y^2}\left[\ln\frac{y}{x} + \frac{25}{12}\right] + O\!\left(\frac{x^4}{y^4}\right)\right) \tag{37}$$

Using this result, it can be shown that, as $b \to \infty$, an approximation of L'_{DC} given by (32) and (33) for the configuration of Fig. 1 is

$$L'_{DC} = \ell'(t,w) + \frac{\mu_0}{4\pi}\left(2\ln\frac{b}{4L_0} + 1\right) + o(1) \tag{38}$$

The curve A of Fig. 5 shows the exact value of L'_{DC} computed as a function of b for $t = h = a = w = 50$ µm, using (32) and (33). The curve B of Fig. 5 shows the approximate value of L'_{DC} given by (38) for the same configuration, which provides an oblique asymptote of the curve A in the semi-log plot.

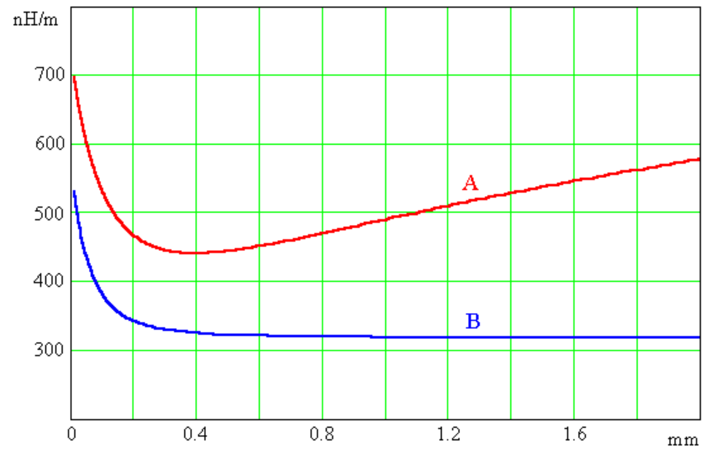

Fig. 3. For the first example, dc p.u.l. inductance L'_{DC} (curve A) and h.f. p.u.l. external inductance L'_0 (curve B) as a function of b.

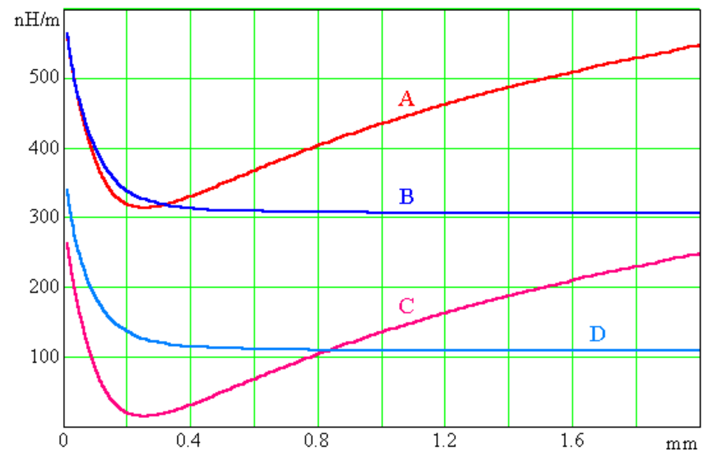

Fig. 4. For the second example, $L'_{DC\,11}$ (curve A), $L'_{0\,11}$ (curve B), $L'_{DC\,12}$ (curve C) and $L'_{0\,12}$ (curve D) as a function of b.

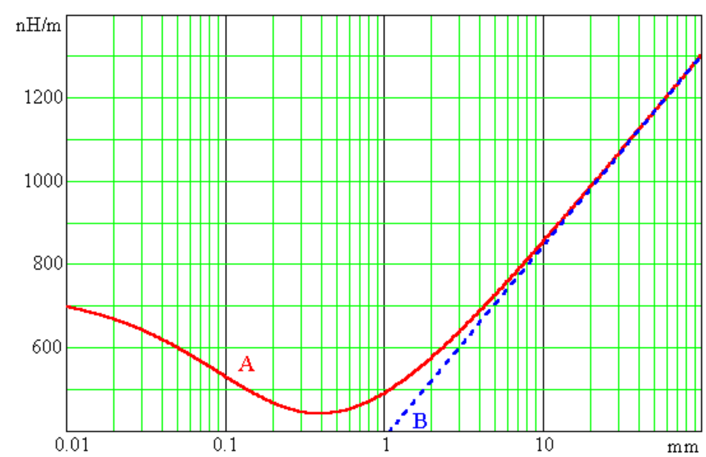

Fig. 5. For the first example, exact dc p.u.l. inductance L'_{DC} (curve A) and the approximation for large b (curve B), as a function of b.

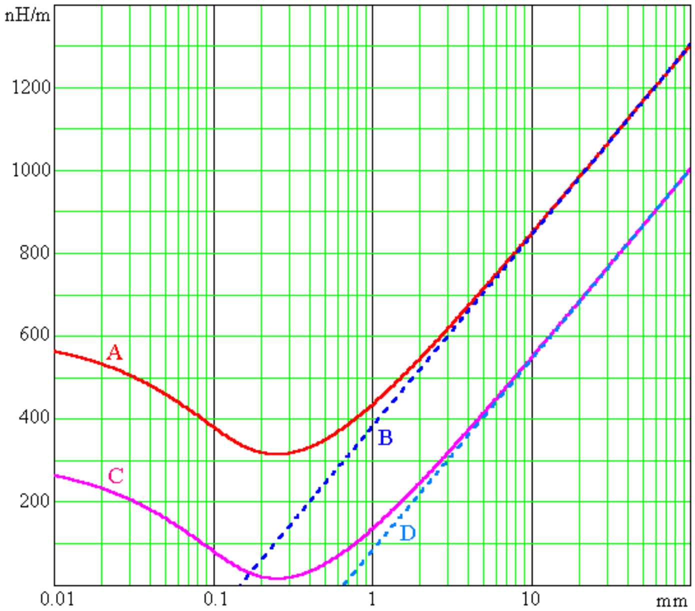

Fig. 6. For the second example, exact $L'_{DC\,11}$ (curve A) and $L'_{DC\,12}$ (curve C) and the corresponding approximations for large b (curve B and curve D, respectively), as a function of b.

Using (37), it can also be shown that, as $b \to \infty$, an approximation of the entries $L'_{DC\,11} = L'_{DC\,22}$ and $L'_{DC\,12} = L'_{DC\,21}$ given by (34) to (36) for the configuration of Fig. 2 is

$$L'_{DC\,11} = \ell'(t,w) + \frac{\mu_0}{4\pi}\left(2\ln\frac{b}{4\mathcal{L}_0} + 1\right) + o(1) \tag{39}$$

and

$$L'_{DC\,12} = m'_{12} + \frac{\mu_0}{4\pi}\left(2\ln\frac{b}{4\mathcal{L}_0} + 1\right) + o(1) \tag{40}$$

The curves A and C of Fig. 6 show the exact values of $L'_{DC\,11}$ and $L'_{DC\,12}$, respectively, as a function of b for $t = h = a = w = d =$ 50 µm, using (34) to (36). The curves B and D of Fig. 6 show the corresponding approximate values given by (39) and (40) for the same configuration, which provide oblique asymptotes of the curves A and C in the semi-log plot.

VII. Conclusion

We have introduced the new concept of modified partial inductances of parallel cylindrical conductors, which can be computed for any cross-section of the conductors and used to directly obtain L'_{DC} or $\mathbf{L}'_{DC}$. In the special case where this cross-section is a set of rectangles having an horizontal side, we have provided exact analytical expressions for the modified partial inductances, so that the exact L'_{DC} or $\mathbf{L}'_{DC}$ can be easily obtained. For two examples, we have shown that L'_{DC} and all entries of $\mathbf{L}'_{DC}$ become large and are equivalent to $\mu_0 \ln b/(2\pi)$ as $b \to \infty$. Thus, L'_{DC} and $\mathbf{L}'_{DC}$ are only defined for a GC of finite width. We have also obtained a more accurate asymptotic expansion of L'_{DC} and $\mathbf{L}'_{DC}$, for a wide reference conductor.

The fact that dc p.u.l. inductances become large as $b \to \infty$ is surprising because it differs so much from the behavior of h.f. p.u.l. inductances. This fact follows from an homogeneous dc current distribution in the GC, for which most of the current in the GC flows at a distance of the TC which increases with b.

An analytical and inherently passive model for the p.u.l. impedance matrix of a MTL has recently been introduced [17] [18]. The model parameters which characterize the high-frequency behavior of the MTL can be obtained at a low computational cost, using an accurate h.f. current distribution which takes into account the skin effect, the edge effect and the proximity effect. This model also comprises other parameters which determine the behavior of the MTL at low frequencies. We plan to use the results of the present paper to clarify the meaning of these parameters.

References

[1] F. Broydé, E. Clavelier, *Tutorial on Echo and Crosstalk in Printed Circuit Boards and Multi-Chip Modules - Lecture Slides*, 2nd Edition, Excem, February 2012, ISBN 978-2-909056-06-7.

[2] E. Rosa, “The self and mutual inductance of linear conductors”, *Bulletin of the Bureau of Standards*, Vol. 4, No. 2, pp. 301-344, 1907.

[3] E. Rosa, F.W. Grover, “Formulas and Tables for the Calculation of Mutual and Self-Inductances”, Third Edition, *Bulletin of the Bureau of Standards*, Vol. 8, No. 1, pp. 1-248, 1916.

[4] F.W. Grover, *Inductance Calculations*, Instrument Society of America, 1973, original publication: Van Nostrand, 1946.

[5] C.T.A. Johnk, *Engineering Electromagnetic Fields and Waves*, John Wiley & Sons, 1975.

[6] W.R. Smythe, *Static and Dynamic Electricity*, third edition, revised printing, Hemisphere Publishing, 1989.

[7] C. Hoer, C. Love, “Exact Inductance Equations for Rectangular Conductors With Applications to More Complicated Geometries”, *J. Res. Nat. Bur. Stand.*, Vol. 69C, No. 2, pp. 127-137, April-June 1965.

[8] A. E. Ruehli, “Inductance Calculations in a Complex Integrated Circuit Environment”, *IBM J. Res. Develop.*, pp. 470-480, Sept. 1972.

[9] R.-B. Wu, C.-N. Kuo, K. K. Chang, “Inductance and Resistance Computation for Three-Dimensional Multiconductor Interconnection Structures”, *IEEE Trans. Microwave Theory Tech.*, Vol. 40, No. 2, pp. 263-271, Feb. 1992.

[10] C.R. Paul, “What do we mean by «Inductance»? — Part II: Partial Inductance”, *IEEE EMC Society Newsletter*, No. 216, pp. 72-79, Winter 2008.

[11] C.R. Paul, “Partial Inductance”, *IEEE EMC Society Newsletter*, No. 226, pp. 34-41, Summer 2010.

[12] G. Zuong, C.-K. Koh, “Exact Closed-Form Formula for Partial Mutual Inductances of Rectangular Conductors”, *IEEE Trans. Circuits Syst.*, Vol. 50, No. 10, pp. 1349-1353, Oct. 2003.

[13] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Unabridged Third edition, Volume Two, Clarendon Press, 1891.

[14] A. Gray, *The theory and practice of absolute measurements in electricity and magnetism*, Vol. II, Part I, Macmillan and co., 1893.

[15] E. Rosa, “On the geometrical mean distances of rectangular areas and the calculation of self-inductance”, *Bulletin of the Bureau of Standards*, Vol. 3, No. 1, pp. 1-41, 1907.

[16] F. Broydé, E. Clavelier, “A Computation of the High-Frequency Per-Unit-Length Resistance Matrix of a Multiconductor Interconnection”, *Proc. of the 10th Int. Symp. on Electromagnetic Compatibility, EMC Europe 2011*, pp. 351-356, Sept. 26-30, 2011.

[17] F. Broydé, E. Clavelier, “A Passive Analytical Per-Unit-Length Internal Impedance Matrix Model for Multiconductor Interconnections”, *Proc. 2011 IEEE Int. Symp. on Electromagnetic Compatibility, EMC 2011*, Aug. 2011.

[18] F. Broydé, E. Clavelier, “An Analytical Resistive Loss Model for Multiconductor Transmission Lines and the Proof of its Passivity”, *Proc. IEEE 20th Conference on Electrical Performance of Electronic Packaging and Systems, EPEPS 2011*, Oct. 23-26, 2011.