Per-Unit-Length Inductance Matrix Computations Using Modified Partial Inductances

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1. Introduction

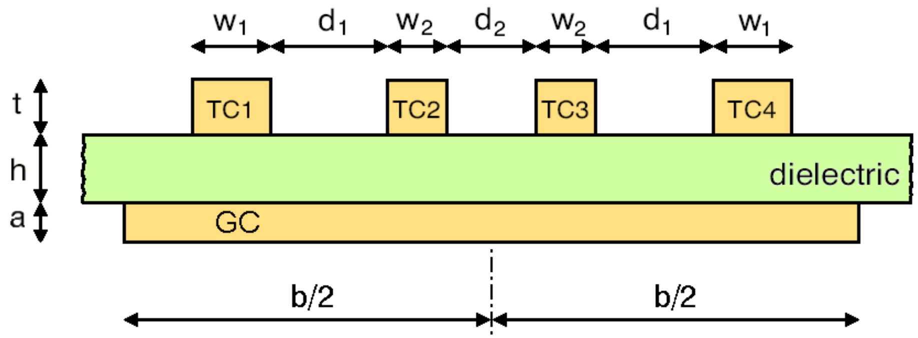
❒ We consider a uniform multiconductor interconnection having n TCs and a GC.

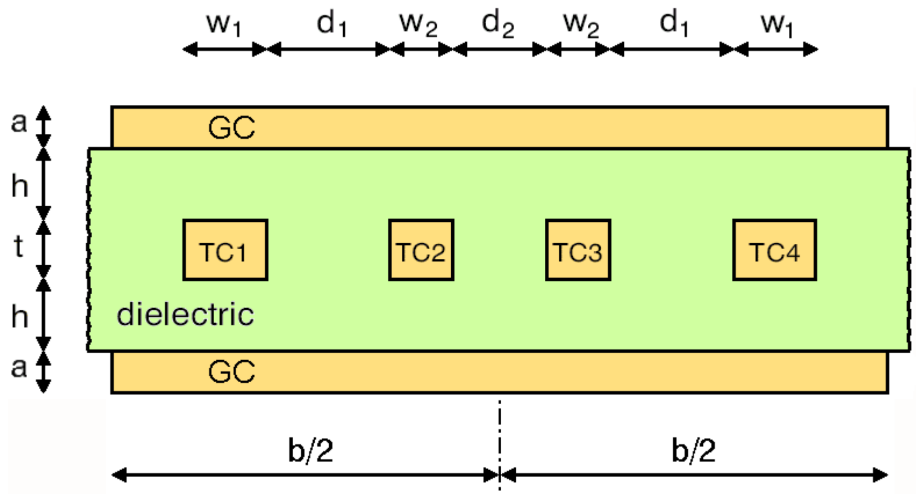
❒ A parameter of the MTL model is $\mathbf{Z}'$. For $f < f_o$ we have

$$\mathbf{Z}' \approx \mathbf{R}'_{DC} + 2\pi f \mathbf{L}'_{DC} \qquad (1)$$

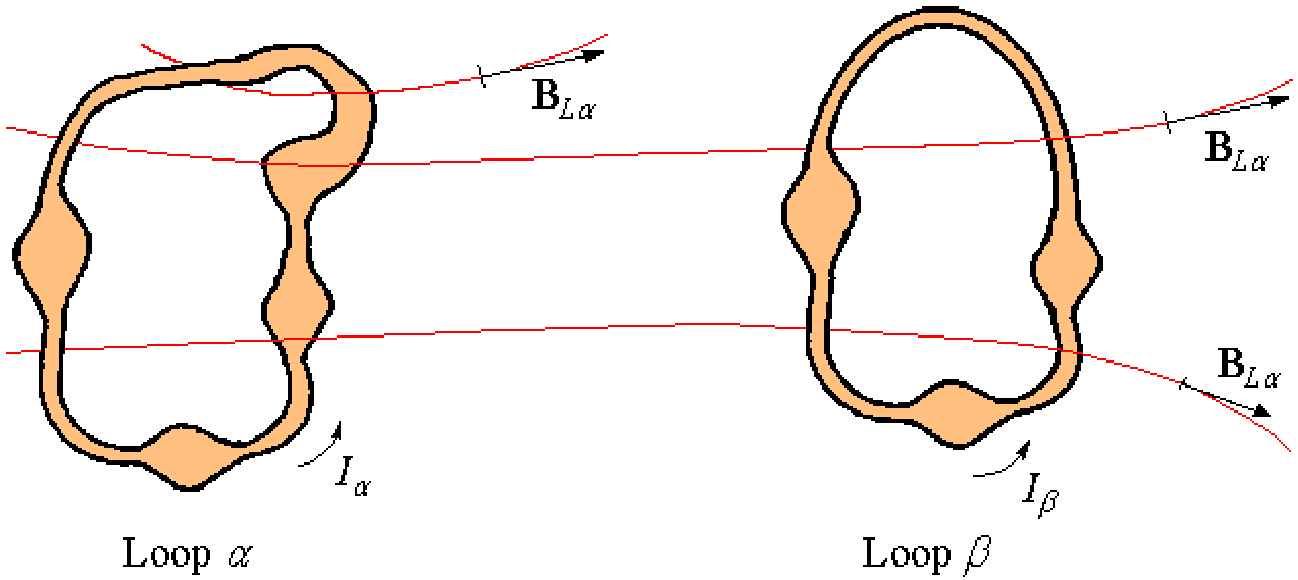
❒ This paper is about a new approach for the computation of $\mathbf{L}'_{DC}$.

❒ We will use the multiconductor microstrip and the multiconductor stripline as example, with $t = h = a = w_1 = w_2 = d_1 = d_2 = 50$ µm.





2. Partial inductance revisited



❒ Assuming the conservation of current in each loop,

- I_α is the current in the loop α and I_β is the current in the loop β ;
- $\mathbf{B}_{L\alpha}$ and $\mathbf{H}_{L\alpha}$ are the fields produced anywhere in space by I_α .

❒ We define the self-inductances and the mutual inductances using

$$L_{DC\alpha\beta} I_\alpha I_\beta = \iiint_V \mathbf{B}_{L\alpha} \cdot \mathbf{H}_{L\beta} \, dv \tag{2}$$

❒ If we consider the branches 1,..., N forming the loops, let us use:

- ◆ i_α to denote the current in the branch α ;
- ◆ $\mathbf{B}_{b\alpha}$ and $\mathbf{H}_{b\alpha}$ are the fields produced anywhere in space by i_α .

❒ For the dc current distribution, we define the partial self-inductances and the partial mutual inductances using

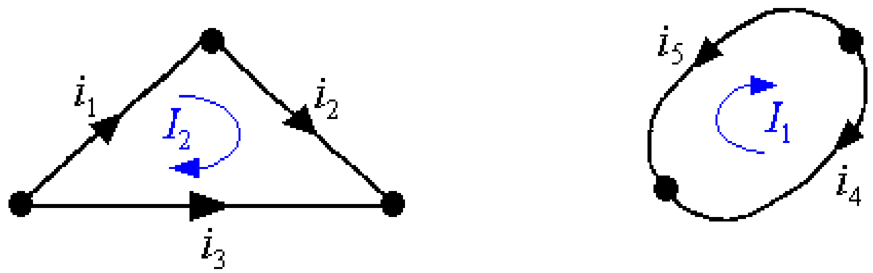
$$m_{\alpha\beta} i_\alpha i_\beta = \iiint_V \mathbf{B}_{b\alpha} \cdot \mathbf{H}_{b\beta} \, dv \tag{3}$$

❒ A loop α is formed by the branches of the subset $N_\alpha \subset \{1,..., N\}$. For a branch $p \in N_\alpha$, let us define $\varepsilon_\alpha(p)$ by: $\varepsilon_\alpha(p) = 1$ if the branch p and the loop α have the same reference direction, $\varepsilon_\alpha(p) = -1$ otherwise. For computing the dc inductance matrix $\mathbf{L}_{DC} = [L_{DC\,\alpha\,\beta}]$, we can use known partial inductance and

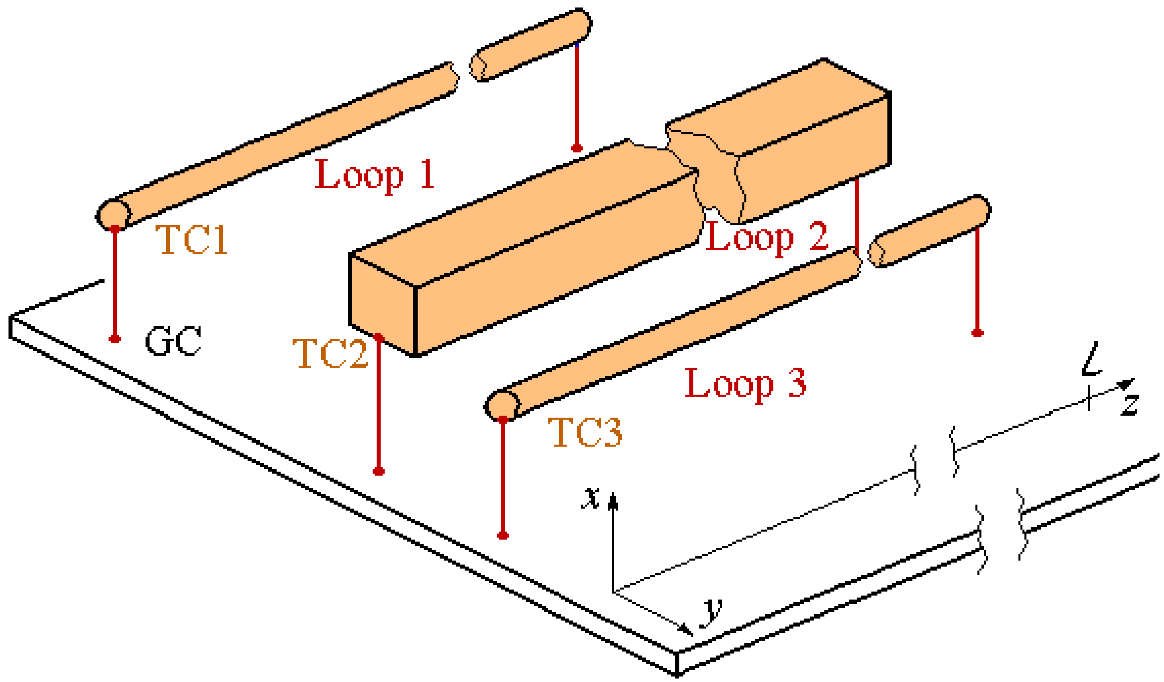
$$L_{DC\alpha\beta} = \sum_{p \in N_\alpha} \sum_{q \in N_\beta} \varepsilon_\alpha(p)\varepsilon_\beta(q) m_{pq} \qquad (4)$$

❒ The proof of (4) uses the fact that the current distributions are independent of each other.

❒ Example: 2 loops, 5 branches.



$$L_{DC\,11} = m_{44} + m_{55} - 2\,m_{45}$$
$$L_{DC\,22} = m_{11} + m_{22} + m_{33} + 2\,m_{12} - 2\,m_{13} - 2\,m_{23}$$
$$L_{DC\,12} = L_{DC\,21} = m_{14} - m_{15} + m_{24} - m_{25} - m_{34} + m_{35}$$



❐ For a $(n+1)$-conductor uniform MTL, for a uniform current distribution and for ℓ >> transverse dimensions, $\mathbf{L}_{DC}$ is nearly proportional to ℓ. The p.u.l. inductance matrix is

$$\mathbf{L}'_{DC} = \lim_{\ell \to \infty} \frac{\mathbf{L}_{DC}}{\ell} \tag{5}$$

❐ $\mathbf{L}_{DC}$ and $\mathbf{L}'_{DC}$ are positive definite real symmetric $n \times n$ matrices.

3. Modified partial inductance

❒ At this stage, to obtain the dc p.u.l. inductance matrix $\mathbf{L}'_{DC}$ of an interconnection made of parallel straight conductors, we can compute $\mathbf{L}_{DC}$ versus $\mathcal{L}$ using partial inductances, and then apply (5).

❒ This route is strange and it leads to numerical problems.

❒ To avoid it, we define the *modified partial inductance* of the parallel conductors α and β, denoted by $m'_{\alpha\beta}$, as

$$m'_{\alpha\beta} = \lim_{\mathcal{L}\to\infty}\left(\frac{m_{\alpha\beta}}{\mathcal{L}} - \frac{\mu_0}{2\pi}\ln\frac{2\mathcal{L}}{\mathcal{L}_0}\right) \tag{6}$$

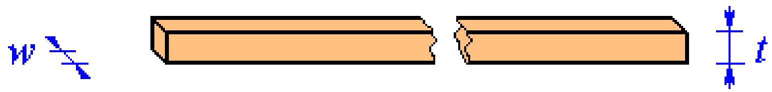
where $\mathcal{L}_0$ is an arbitrary length, which must be the same for all modified partial inductances used in the same formula.

❒ To obtain the dc p.u.l. inductance matrix $\mathbf{L}'_{DC} = [L_{DC\,\alpha\,\beta}]$, we can use known modified partial inductances and

$$L'_{DC\alpha\beta} = \sum_{p \in N'_\alpha} \sum_{q \in N'_\beta} \varepsilon_\alpha(p)\varepsilon_\beta(q)\, m'_{pq} \qquad (7)$$

where the loop α contains two branches extending from $z = 0$ to $z = \mathcal{L}$, the branches of the subset $N'_\alpha \subset \{1,\ldots, N\}$.

❒ Modified partial self-inductance of a conductor of rectangular cross section:



$$m'_{\alpha\alpha} = \frac{\mu_0}{4\pi}\left(-\ln\frac{t^2 + w^2}{\mathcal{L}_0^{\,2}} - \frac{4}{3}\left\{ \frac{t}{w}\tan^{-1}\frac{w}{t} + \frac{w}{t}\tan^{-1}\frac{t}{w} \right\} + \frac{1}{6}\left\{ \frac{t^2}{w^2}\ln\left(1 + \frac{w^2}{t^2}\right) + \frac{w^2}{t^2}\ln\left(1 + \frac{t^2}{w^2}\right) \right\} + \frac{13}{6} \right) \qquad (8)$$

❒ Modified partial mutual inductance of conductors of rectangular cross section:

The cross-section of the conductor α extending from $x = x_\alpha$ to $x = x_\alpha + t_\alpha$ and from $y = y_\alpha$ to $y = y_\alpha + w_\alpha$, where $w_\alpha > 0$ and $t_\alpha > 0$, $m'_{\alpha\beta}$ is given by

$$m'_{\alpha\beta} = \frac{\sum_{I=1}^{2}\sum_{J=1}^{2}\sum_{L=1}^{2}\sum_{M=1}^{2}(-1)^{I+J+L+M}\left(X_{\alpha I} - X_{\beta L}\right)^2\left(Y_{\alpha J} - Y_{\beta M}\right)^2 m'_{I,J,L,M}}{4\, t_\alpha\, t_\beta\, w_\alpha\, w_\beta} \qquad (9)$$

where

$$\mathbf{X}_\alpha = \begin{pmatrix} x_\alpha \\ x_\alpha + t_\alpha \end{pmatrix} \qquad \mathbf{Y}_\alpha = \begin{pmatrix} y_\alpha \\ y_\alpha + w_\alpha \end{pmatrix} \qquad (10)$$

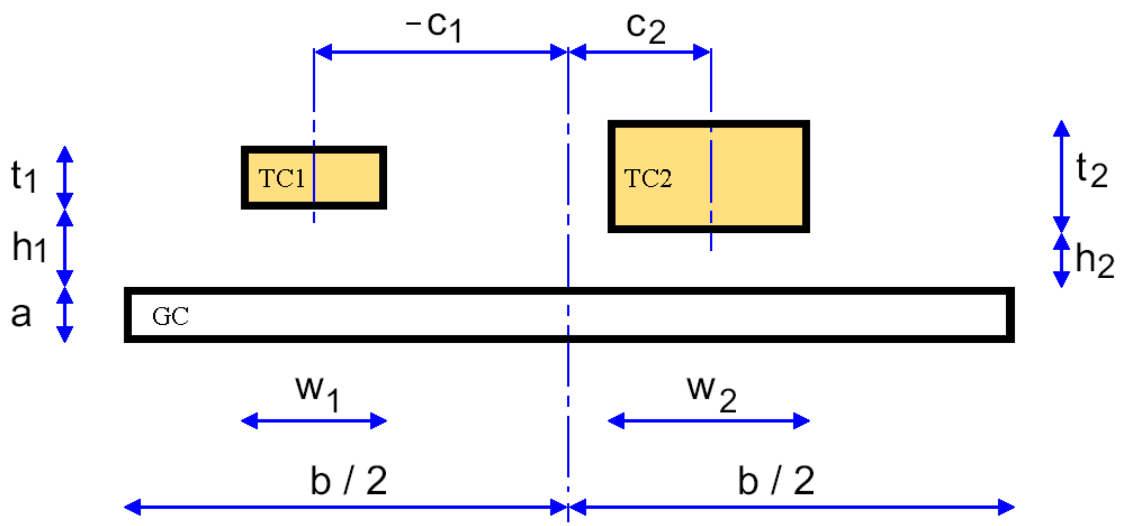
and

$$m'_{I,J,L,M} = \begin{cases} 0 \text{ if } \left(Y_{\alpha J} - Y_{\beta M}\right)\left(X_{\alpha I} - X_{\beta L}\right) = 0 \\ \ell'\left(\left|Y_{\alpha J} - Y_{\beta M}\right|, \left|X_{\alpha I} - X_{\beta L}\right|\right) \text{ else} \end{cases} \qquad (11)$$

where $\ell'(y, x)$ is the modified partial self-inductance of a conductor of uniform rectangular cross-section of width y and thickness x, given by (8).

4. Computation of p.u.l. inductance matrices

❒ This configuration can be used to compute the $L_{DC\,\alpha\,\beta}$ of any interconnection having a GC made of a single rectangular conductor.



❒ For this problem, $\mathbf{L}'_{DC}$ is exactly given by

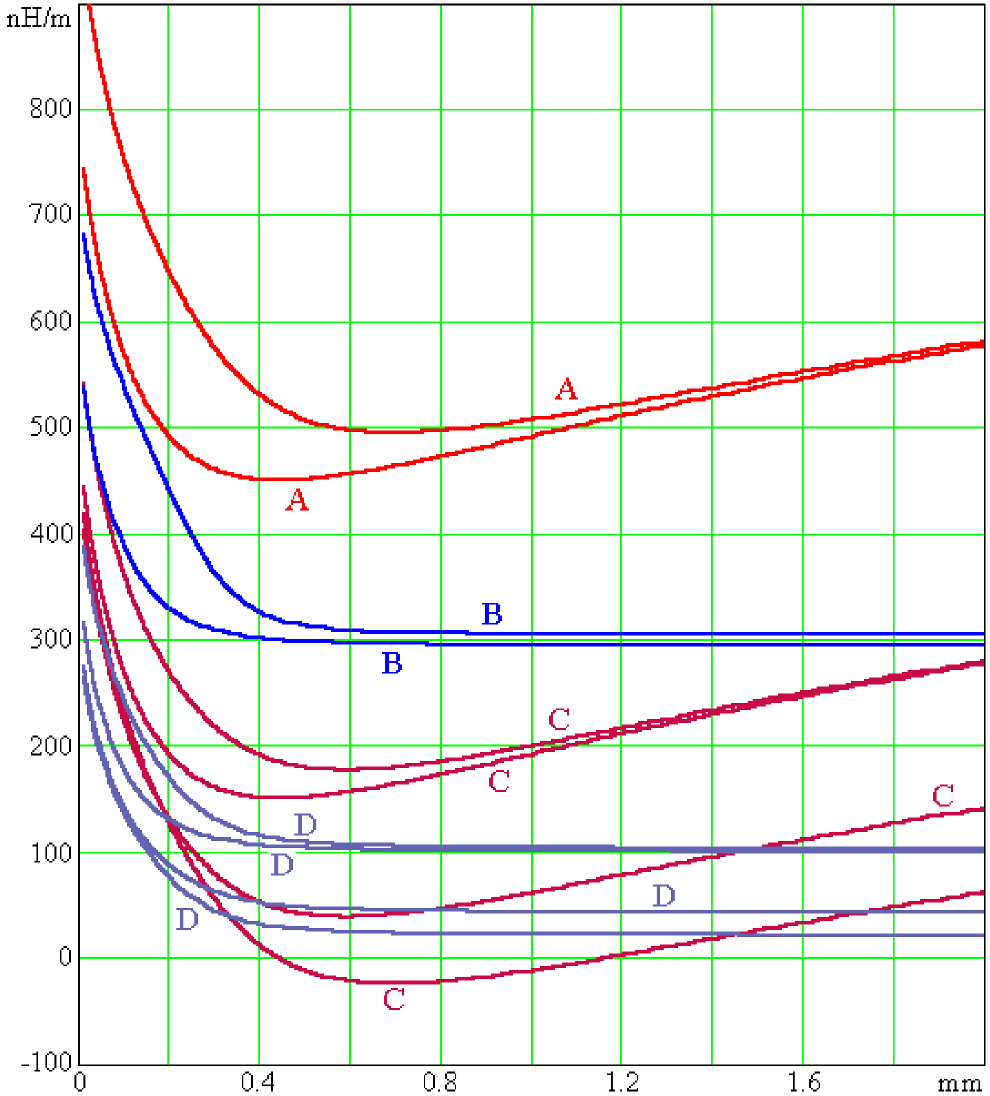
$$L'_{DC\alpha\alpha} = m'_{\alpha\alpha} + m'_{33} - 2m'_{\alpha 3} \tag{12}$$

and

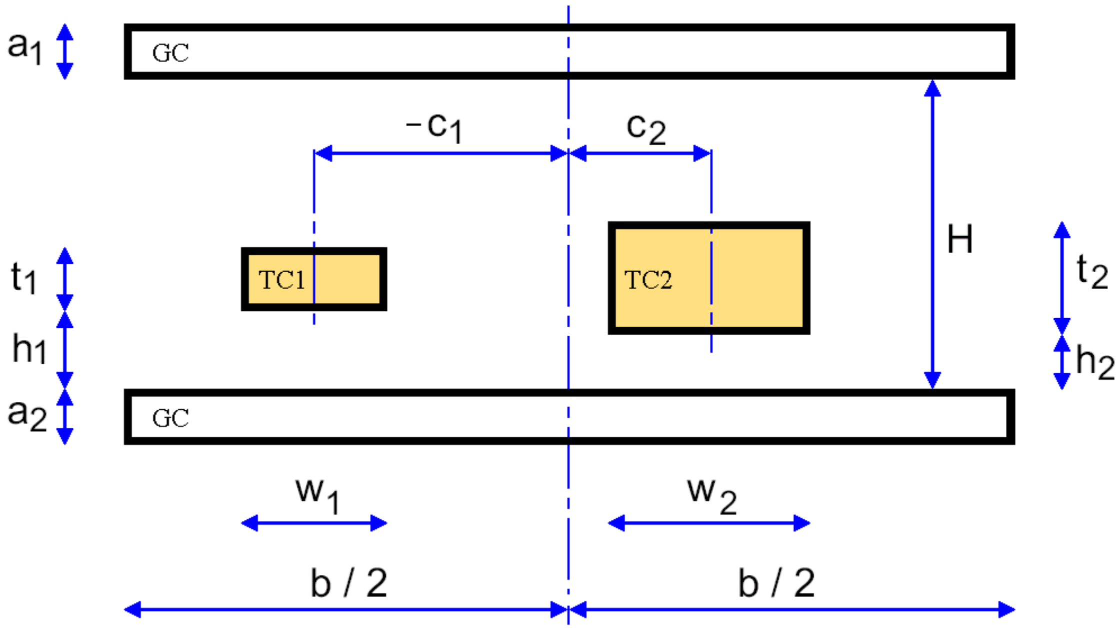
$$L'_{DC12} = L'_{DC21} = m'_{12} - m'_{13} - m'_{23} + m'_{33} \tag{13}$$

❒ It is interesting to compare $\mathbf{L}'_{DC}$ with the high-frequency p.u.l. external inductance matrix, denoted by $\mathbf{L}'_0$.

The figure shows the entries of $\mathbf{L}'_{DC}$ and $\mathbf{L}'_0$, computed as a function of b, for the multiconductor microstrip defined in the introduction: the diagonal entries of $\mathbf{L}'_{DC}$ (2 curves A), the diagonal entries of $\mathbf{L}'_0$ (2 curves B), the non-diagonal entries of $\mathbf{L}'_{DC}$ (4 curves C) and the non-diagonal entries of $\mathbf{L}'_0$ (4 curves D).



❐ This configuration can be used to compute the $L_{DC\,\alpha\,\beta}$ of any interconnection having a GC made of two superimposed and identical rectangular conductors.



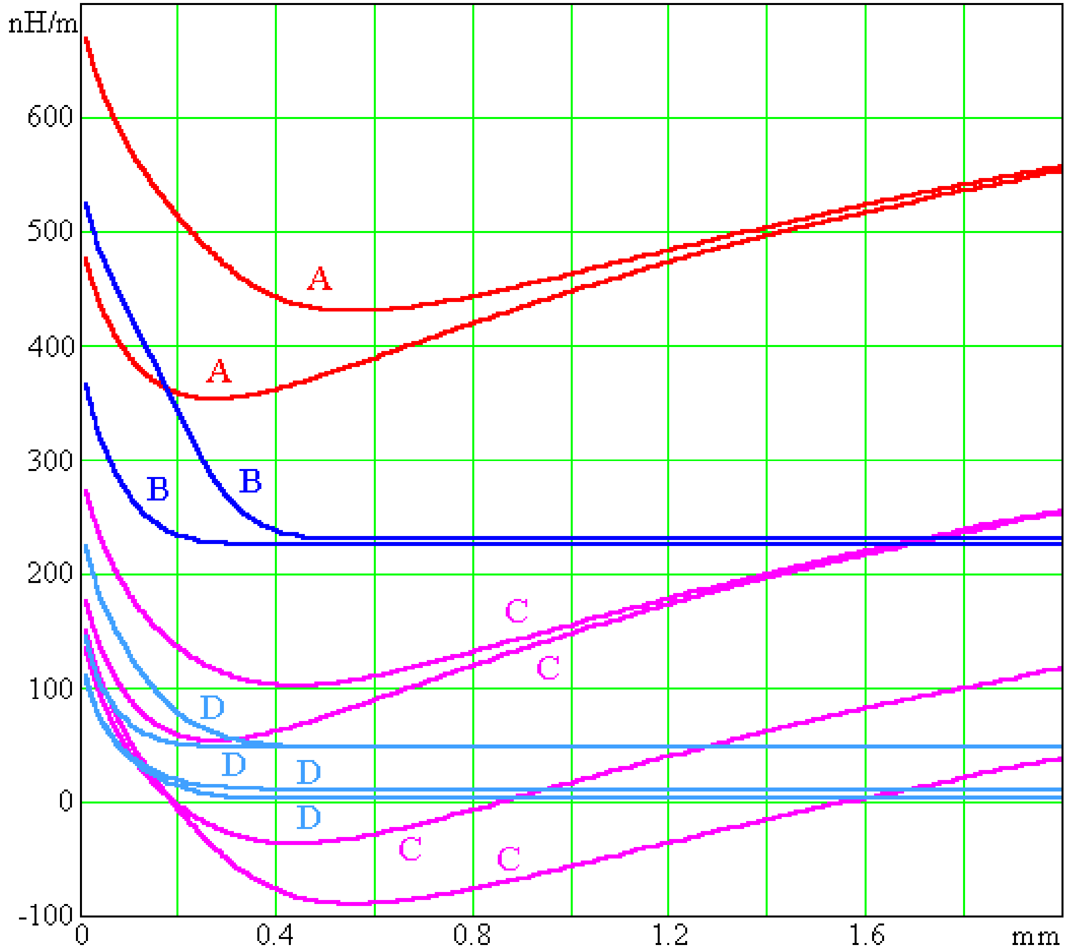
❐ For this problem, $\mathbf{L}'_{DC}$ is exactly given by

$$L'_{DC\alpha\alpha} = m'_{\alpha\alpha} - m'_{\alpha 3} - m'_{\alpha 4} + \frac{m'_{33} + m'_{44} + 2m'_{34}}{4} \qquad (14)$$

and

$$L'_{DC12} = m'_{12} - \frac{m'_{13} + m'_{23} + m'_{14} + m'_{24}}{2} + \frac{m'_{33} + m'_{44} + 2m'_{34}}{4} \qquad (15)$$

The figure shows the entries of $\mathbf{L}'_{DC}$ and $\mathbf{L}'_0$, computed as a function of b, for the multiconductor stripline defined in the introduction: the diagonal entries of $\mathbf{L}'_{DC}$ (2 curves A), the diagonal entries of $\mathbf{L}'_0$(2 curves B), the non-diagonal entries of $\mathbf{L}'_{DC}$ (4 curves C) and the non-diagonal entries of $\mathbf{L}'_0$ (4 curves D).



5. Asymptotic expansions for a broad ground plane

- We now want to explore the behavior of $\mathbf{L}'_{DC}$ as $b \to \infty$.

- For the generic multiconductor microstrip configuration, we obtain:

$$L'_{DC\alpha\beta} = L'_{DC\beta\alpha} = m'_{\alpha\beta} + \frac{\mu_0}{4\pi}\left[2\ln\frac{b}{4\mathcal{L}_0} + 1 + \frac{E_{\alpha\beta}}{b} + \frac{a^2}{3b^2}\ln\frac{b}{a} + \frac{F_{\alpha\beta}}{b^2}\right] + O\left(\frac{1}{b^3}\right) \qquad (16)$$

where

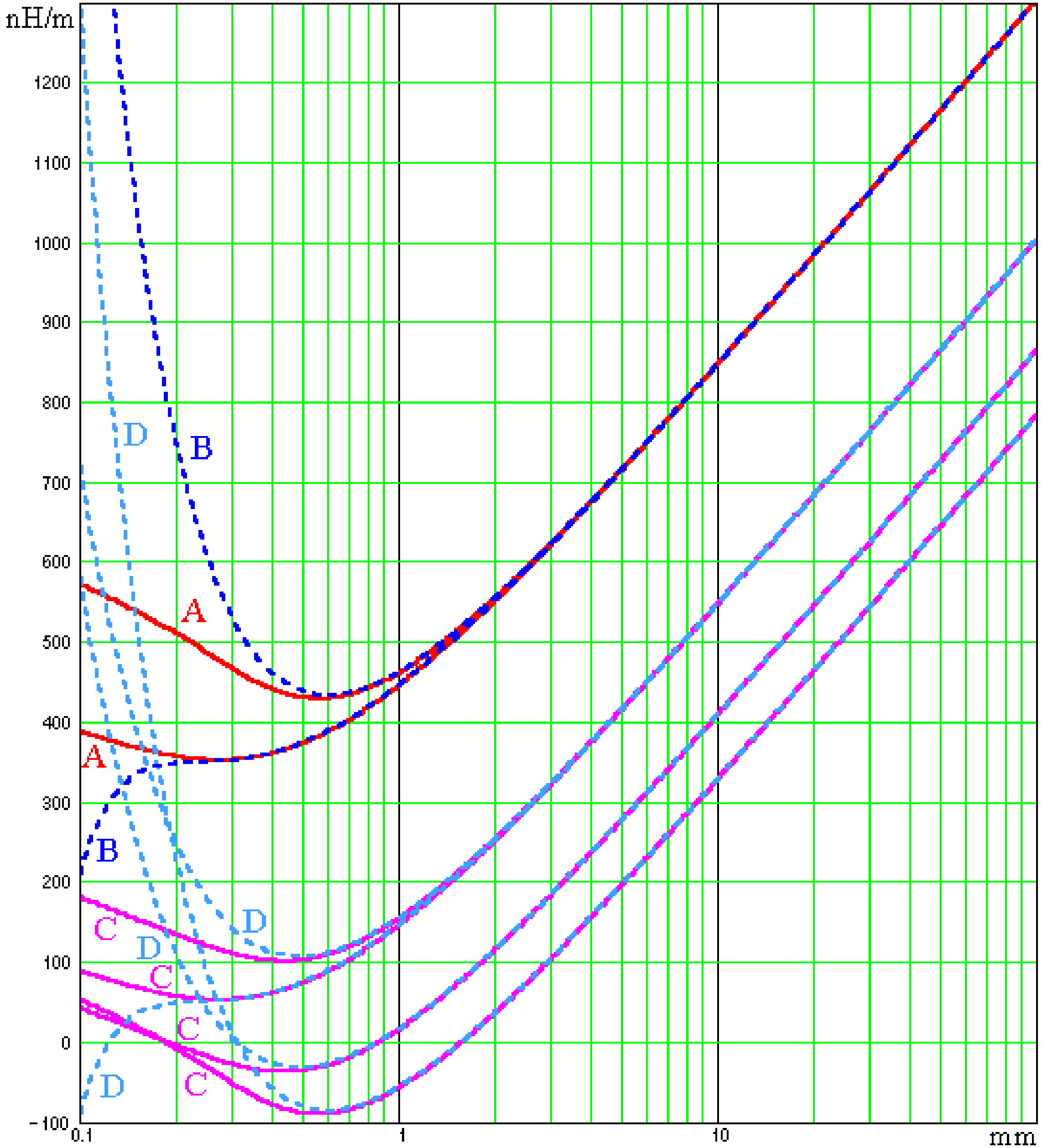
$$\begin{cases} E_{\alpha\beta} = \pi\,\dfrac{4a + 3t_\alpha + 6h_\alpha + 3t_\beta + 6h_\beta}{3} \\ F_{\alpha\beta} = -\dfrac{71a^2}{36} \\ \quad -\dfrac{12h_\alpha^2 + 12(a + t_\alpha)h_\alpha + 6at_\alpha + 4t_\alpha^2 - w_\alpha^2 - 12c_\alpha^2}{3} \\ \quad -\dfrac{12h_\beta^2 + 12(a + t_\beta)h_\beta + 6at_\beta + 4t_\beta^2 - w_\beta^2 - 12c_\beta^2}{3} \end{cases} \qquad (17)$$

❒ For the generic multiconductor stripline configuration we get:

$$L'_{DC\alpha\beta} = m'_{\alpha\beta} + \frac{\mu_0}{4\pi}\left[\begin{array}{l} 2\ln\dfrac{b}{4\mathcal{L}_0} + 1 + \dfrac{U}{b} + \dfrac{a^2}{6b^2}\ln\dfrac{b}{a} + \dfrac{(H+2a)^4}{12a^2b^2}\ln\dfrac{b}{H+2a} \\ + \dfrac{H^4}{12a^2b^2}\ln\dfrac{b}{H} - \dfrac{(H+a)^4}{6a^2b^2}\ln\dfrac{b}{H+a} + \dfrac{V_{\alpha\beta}}{b^2} \end{array}\right] + O\left(\frac{1}{b^3}\right) \qquad (18)$$

where

$$\begin{cases} U = \pi\,\dfrac{2a+3H}{3} \\ V_{\alpha\beta} = \dfrac{75H^2 + 150aH + 4a^2}{36} \\ \quad -\dfrac{1}{3}\left(6(H - h_\alpha - t_\alpha)^2 + 6h_\alpha^2 + 6(a+t_\alpha)(H-t_\alpha) + 6at_\alpha + 4t_\alpha^2 - w_\alpha^2 - 12c_\alpha^2\right) \\ \quad -\dfrac{1}{3}\left(6(H - h_\beta - t_\beta)^2 + 6h_\beta^2 + 6(a+t_\beta)(H-t_\beta) + 6at_\beta + 4t_\beta^2 - w_\beta^2 - 12c_\beta^2\right) \end{cases} \qquad (19)$$



❒ By (16) and (18), all entries of $\mathbf{L}'_{DC}$ are equivalent to $(\mu_0 /(2\pi)) \ln b$ as $b \to \infty$. This corresponds to an oblique asymptote in a semi-log plot, presenting a slope of about 461nH per decade of b.

For the multiconductor stripline, the diagonal entries of $\mathbf{L}'_{DC}$ (2 curves A), their asymptotic expansions (2 curves B), the non-diagonal entries of $\mathbf{L}'_{DC}$ (4 curves C) and their asymptotic expansions (4 curves D) as a function of b.

6. Conclusion

❒ Modified partial inductances can be computed for any cross-section of the conductors and used to directly obtain $\mathbf{L}'_{DC}$.

❒ In the special case where this cross-section is a set of rectangles having an horizontal side, we have provided exact analytical expressions for them.

❒ We have obtained exact analytical expressions for the entries of $\mathbf{L}'_{DC}$ in the cases of a generic microstrip configuration and a generic stripline configuration.

❒ $\mathbf{L}'_{DC}$ may have negative non-diagonal entries, in a range of values of b.

❒ We have computed accurate asymptotic expansions for large values of b, for both generic configurations.

❒ $\mathbf{L}'_{DC}$ is only defined for a finite b, because all entries of $\mathbf{L}'_{DC}$ are equivalent to $(\mu_0 /(2\pi)) \ln b$ as $b \to \infty$.