About the Beam Cosines and the Radiation Efficiency of a Multiport Antenna Array

Frédéric Broydé, Evelyne Clavelier
Excem, Maule, France (emails: fredbroyde@eurexcem.com, eclavelier@eurexcem.com)

***Abstract* — The paper establishes some properties of the matrix of the beam cosines of the radiated power of a multiport antenna array (MAA), and of the matrix of the beam cosines of the delivered power, which concerns reception by the MAA. These matrices are related to several quantities used in the literature. Though these matrices are usually different and complex, the paper explains why they may be equal and real, under certain assumptions. The paper also proposes a definition and an investigation of the radiation efficiency of the MAA.**

***Index Terms* — Antenna theory, multiport antenna arrays, MIMO radio communication.**

I. Introduction

We consider a multiport antenna array (MAA) having m ports numbered from 1 to m, where $m \geq 2$. The MAA may for instance only comprise m antennas, as shown in Fig. 1(A). It may also include a multiple-antenna-port and multiple-user-port (MAPMUP) antenna tuner [1]-[6] and/or one or more feeders, in addition to the antennas, as shown in Fig. 1(B). We assume that the MAA is linear, but we do not assume reciprocity.

In analytic geometry, direction cosine refers to the cosine of the angle between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, that is the quantity $\mathbf{v}_1 \cdot \mathbf{v}_2 /(\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|)$. By analogy, in this paper, some quantities in the form $<\psi_1, \psi_2>/(<\psi_1, \psi_1><\psi_2, \psi_2>)^{1/2}$, where the brackets denote an hermitian product involving an integration over all directions in space, are called "beam cosines" [7, ch. 12].

In [6], we computed several beam cosines of an MAA comprising a MAPMUP antenna tuner. However, we did not explain some visible characteristics of their computed values, and, to our best knowledge, the literature does not help to understand them. This paper provides a new analysis of the properties of the beam cosines, in § II and § III. The paper also proposes a new definition of the radiation efficiency of the MAA, and derives some of its attributes in § IV.

II. Beam Cosines of the Radiated Power

We will use a spherical coordinate system having an origin somewhere close to the MAA, θ being the zenithal angle (i.e. the angle with respect to the z-axis) and φ being the azimuth angle. In this § II, the MAA is used for emission. A linear multiport source (LMS) has m ports numbered from 1 to m. Its admittance matrix is denoted by $\mathbf{Y}_S$. We do not assume that $\mathbf{Y}_S$ is diagonal. For any $\alpha \in \{1,..., m\}$, port α of the MAA is coupled to port α of the LMS.

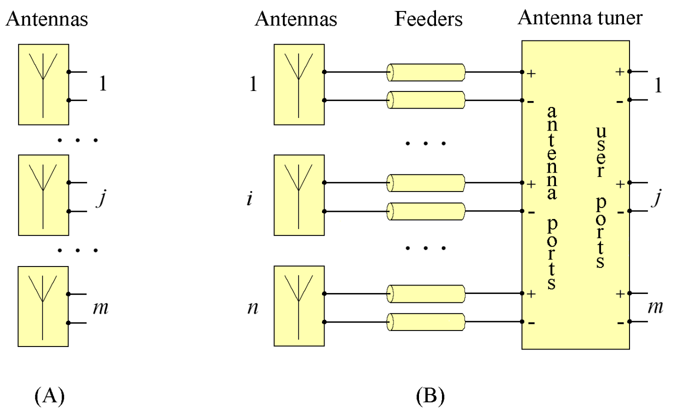

Fig. 1. An MAA consisting of m antennas in (A), and an MAA consisting of n antennas, their feeders and a MAPMUP antenna tuner having n antenna ports and m user ports (B).

Let $\mathbf{I}_0$ be the $m \times 1$ column vector of the rms short-circuit currents at the ports of the LMS. For $\alpha \in \{1,..., m\}$, let $\mathbf{I}_{0\,\alpha}$ be a particular value of $\mathbf{I}_0$ having all its entries equal to zero ampere, except the entry of the row α, which takes on a specified value I_α. For $\alpha \in \{1,..., m\}$, let $\mathbf{E}_{0\,\alpha}$ be the electric field radiated by the MAA in a configuration where $\mathbf{I}_0$ is equal to $\mathbf{I}_{0\,\alpha}$. Clearly, $\mathbf{E}_{0\,\alpha}$ is proportional to I_α. A plot of the average radiation intensity of $\mathbf{E}_{0\,\alpha}$ in the far field, as a function of an angle, may be referred to as a radiation pattern of port α.

Let $\mathbf{P}_R$ be the matrix of the self- and cross complex powers radiated by the MAA over all values of θ and φ, defined as follows: for $\alpha \in \{1,..., m\}$ and $\beta \in \{1,..., m\}$, the entry $P_{R\,\alpha\beta}$ of $\mathbf{P}_R$ is given by

$$P_{R\,\alpha\beta} = \frac{1}{\eta_0} \int_0^{\pi} \int_0^{2\pi} \mathbf{E}_{0\alpha}^{*} \mathbf{E}_{0\beta}\, r^2 \sin\theta \; d\varphi \, d\theta \tag{1}$$

where $\eta_0 \approx 376.7\ \Omega$ is the intrinsic impedance of free space, where $\mathbf{E}_{0\,\alpha}$ and $\mathbf{E}_{0\,\beta}$ are regarded as column vectors, where the star denotes the Hermitian adjoint, and where the integration is carried out at a large distance r from the antennas lying in free space. Clearly, $P_{R\,\alpha\beta}$ is proportional to $\overline{I_\alpha}$ and to I_β, where the bar indicates the complex conjugate.

For $\alpha \in \{1,..., m\}$, since $\mathbf{E}_{0\,\alpha}$ is caused by the short-circuit current I_α of the port α of the LMS, we can consider that $\mathbf{E}_{0\,\alpha}$ is the electric field produced by a single-port antenna made up of the MAA and a linear passive multiport circuit (PMC) having m ports numbered from 1 to m, of admittance matrix $\mathbf{Y}_S$, each port of the PMC being coupled to the port of same number of the MAA, the port of the single-port antenna being port α of the

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MAA coupled to port α of the PMC, this single-port antenna being coupled to a single-port current source delivering I_α. Let $\mathbf{h}_{0\,\alpha}$ be the effective complex length of this single-port antenna in a direction (θ, φ), as defined in [8, § 5.2]. We have

$$\mathbf{E}_{0\alpha} = j\eta_0 \frac{I_\alpha\, k\, e^{-jkr}}{4\pi\, r}\, \mathbf{h}_{0\alpha} \tag{2}$$

where k is the wave number. Consequently, we have

$$P_{R\alpha\beta} = \frac{\eta_0 k^2 \overline{I_\alpha} I_\beta}{16\pi^2} \int_0^\pi \int_0^{2\pi} \mathbf{h}^*_{0\alpha} \mathbf{h}_{0\beta} \sin\theta \; d\varphi\, d\theta \tag{3}$$

Since, for any $\alpha \in \{1,..., m\}$ and any $\beta \in \{1,..., m\}$, by (1), $P_{R\,\alpha\beta}$ is an hermitian product (or inner product) of the beams $\mathbf{E}_{0\beta}$ and $\mathbf{E}_{0\,\alpha}$, we can, if $P_{R\,\alpha\,\alpha} \neq 0$ and $P_{R\,\beta\,\beta} \neq 0$, define the beam cosines of the radiated power, denoted by $\rho_{R\,\alpha\,\beta}$ and given by

$$\rho_{R\alpha\beta} = \frac{P_{R\alpha\beta}}{\sqrt{P_{R\alpha\alpha} P_{R\beta\beta}}} \tag{4}$$

The matrix ($\rho_{R\,\alpha\beta}$) is Hermitian. Note that $|\rho_{R\,\alpha\beta}|$ is sometimes referred to as "orthogonality coefficients", for instance in [9]. Also, $|\,\rho_{R\,\alpha\beta}\,|^2$ is related to the "envelope correlation" considered in [10] and to the "correlation coefficient" considered in [11, pp. 248-249]. We note that $\rho_{R\,\alpha\,\beta}$ may exist only if $I_\alpha \neq 0$ and $I_\beta \neq 0$, that $\rho_{R\,\alpha\,\beta}$ is proportional to $\overline{I_\alpha}/|I_\alpha|$ and to $I_\beta/|I_\beta|$, and that $|\,\rho_{R\,\alpha\,\beta}\,|$ and $|\,\rho_{R\,\alpha\beta}\,|^2$ are independent of I_α and I_β.

For an arbitrary $\mathbf{I}_0$, let $\mathbf{E}_0$ be the electric field radiated by the MAA and let P_{rad} be the average power radiated by the MAA. For any $\alpha \in \{1,..., m\}$, we now use I_α to denote the entry of the row α of $\mathbf{I}_0$. P_{rad} is given by

$$P_{\text{rad}} = \frac{1}{\eta_0} \int_0^\pi \int_0^{2\pi} \mathbf{E}^*_0 \mathbf{E}_0\, r^2 \sin\theta \; d\varphi\, d\theta \tag{5}$$

where $\mathbf{E}_0$ is regarded as a column vector, and where the integration is carried out at a large distance r from the antennas lying in free space. By superposition, we have

$$\mathbf{E}_0 = \sum_{\alpha=1}^{m} \mathbf{E}_{0\alpha} = j\eta_0 \frac{k\, e^{-jkr}}{4\pi\, r} \sum_{\alpha=1}^{m} I_\alpha\; \mathbf{h}_{0\alpha} \tag{6}$$

so that

$$P_{\text{rad}} = \frac{\eta_0 k^2}{16\pi^2} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \overline{I_\alpha} I_\beta \int_0^\pi \int_0^{2\pi} \mathbf{h}^*_{0\alpha} \mathbf{h}_{0\beta} \sin\theta \; d\varphi\, d\theta \tag{7}$$

Let $\mathbf{Z}_{\text{rad}}$ be the matrix such that, for any $\alpha \in \{1,..., m\}$ and any $\beta \in \{1,..., m\}$, the entry of the row α and the column β of $\mathbf{Z}_{\text{rad}}$, denoted by $Z_{\text{rad}\,\alpha\,\beta}$, is given by

$$Z_{\text{rad}\alpha\beta} = \frac{\eta_0 k^2}{16\pi^2} \int_0^\pi \int_0^{2\pi} \mathbf{h}^*_{0\alpha} \mathbf{h}_{0\beta} \sin\theta \; d\varphi\, d\theta \tag{8}$$

We see that $\mathbf{Z}_{\text{rad}}$ is an Hermitian matrix, that it has the dimensions of impedance, and that

$$P_{\text{rad}} = \mathbf{I}^*_0\, \mathbf{Z}_{\text{rad}}\, \mathbf{I}_0 \tag{9}$$

Thus, P_{rad} is an Hermitian quadratic form of the variable $\mathbf{I}_0$, and $\mathbf{Z}_{\text{rad}}$ is the only matrix which satisfies (9) for any value of $\mathbf{I}_0$ [12, § 3.2.4], [13, p. 174 Problem 6]. Moreover, since $P_{\text{rad}} \geq 0$ for any value of $\mathbf{I}_0$, it follows that $\mathbf{Z}_{\text{rad}}$ is positive semidefinite.

If we use $\mathbf{Y}_A$ to denote the admittance matrix presented by the MAA, we find that the average power received by the ports of the MAA, denoted by P_{ANT}, is given by

$$P_{\text{ANT}} = \mathbf{I}^*_0\, \mathbf{Z}_{\text{pow}}\, \mathbf{I}_0 \tag{10}$$

where

$$\mathbf{Z}_{\text{pow}} = \left(\mathbf{Y}_A + \mathbf{Y}_S\right)^{-1*} \frac{\mathbf{Y}_A + \mathbf{Y}^*_A}{2} \left(\mathbf{Y}_A + \mathbf{Y}_S\right)^{-1} \tag{11}$$

is an impedance matrix, and Hermitian. Since $P_{\text{ANT}} \geq 0$ for any value of $\mathbf{I}_0$, it follows that $\mathbf{Z}_{\text{pow}}$ is positive semidefinite. In the special case where $\mathbf{Y}_A$ is symmetric (reciprocal MAA), the ratio ($\mathbf{Y}_A + \mathbf{Y}_A$*)/2 is the real part of $\mathbf{Y}_A$.

It follows from the conservation of average power that we have

$$P_{\text{ANT}} = P_{\text{rad}} + P_{\text{loss}} \tag{12}$$

where P_{loss} is the loss in the MAA. Using (9), (10) and (12), we find that

$$P_{\text{loss}} = \mathbf{I}^*_0\, \mathbf{Z}_{\text{loss}}\, \mathbf{I}_0 \tag{13}$$

where

$$\mathbf{Z}_{\text{loss}} = \mathbf{Z}_{\text{pow}} - \mathbf{Z}_{\text{rad}} \tag{14}$$

is an impedance matrix, which is Hermitian. We see that P_{loss} is an Hermitian quadratic form of the variable $\mathbf{I}_0$, so that $\mathbf{Z}_{\text{loss}}$ is the only matrix which satisfies (13) for any value of $\mathbf{I}_0$. Since $P_{\text{loss}} \geq 0$ for any value of $\mathbf{I}_0$, $\mathbf{Z}_{\text{loss}}$ is positive semidefinite.

For any $\alpha \in \{1,..., m\}$ and any $\beta \in \{1,..., m\}$, let $Z_{\text{pow}\,\alpha\,\beta}$ and $Z_{\text{loss}\,\alpha\,\beta}$ be the entries of the row α and the column β of $\mathbf{Z}_{\text{pow}}$ and $\mathbf{Z}_{\text{loss}}$, respectively. By (8) and (14), we have

$$Z_{\text{pow}\alpha\beta} - Z_{\text{loss}\alpha\beta} = \frac{\eta_0 k^2}{16\pi^2} \int_0^\pi \int_0^{2\pi} \mathbf{h}^*_{0\alpha} \mathbf{h}_{0\beta} \sin\theta \; d\varphi\, d\theta \tag{15}$$

Using (3), we obtain

$$P_{R\alpha\beta} = \left(Z_{\text{pow}\alpha\beta} - Z_{\text{loss}\alpha\beta}\right) \overline{I_\alpha} I_\beta \tag{16}$$

so that, using (4) we get

$$\rho_{R\alpha\beta} = \frac{Z_{\text{pow}\alpha\beta} - Z_{\text{loss}\alpha\beta}}{\sqrt{\left(Z_{\text{pow}\alpha\alpha} - Z_{\text{loss}\alpha\alpha}\right)\left(Z_{\text{pow}\beta\beta} - Z_{\text{loss}\beta\beta}\right)}} \frac{\overline{I_\alpha}}{|I_\alpha|} \frac{I_\beta}{|I_\beta|} \tag{17}$$

Formula (17) is similar to, but more general than formula (4) of [9] about $|\rho_{R\,\alpha\beta}|$. The main application of this formula is in the case where losses are negligible, for which

$$\rho_{R\alpha\beta} \approx \frac{Z_{\text{pow}\alpha\beta}}{\sqrt{Z_{\text{pow}\alpha\alpha} Z_{\text{pow}\beta\beta}}} \frac{\overline{I_\alpha}}{|I_\alpha|} \frac{I_\beta}{|I_\beta|} \tag{18}$$

so that, in this case, using (11), the beam cosines can be derived from $\mathbf{Y}_S$ and a measurement of $\mathbf{Y}_A$. Formula (18) is similar to, but more general than the result obtained in [10] about $|\rho_{R\,\alpha\beta}|$. It entails that, if $\mathbf{Y}_S$ and $\mathbf{Y}_A$ are diagonal, then the beams are orthogonal (an old result [14]), since in this case, for $\alpha \neq \beta$, we have $Z_{\text{pow}\,\alpha\beta} = 0\ \Omega$ so that $\rho_{R\,\alpha\beta} \approx 0$. The orthogonality considered here is "over all directions in space", as opposed to an orthogonality "over the azimuth φ" which does not seem to exist.

Beam cosines of the radiated power were computed in [6, § 6] for an MAA composed of 4 antennas, 4 feeders and a lossy MAPMUP antenna tuner having 4 antenna ports and 4 user ports. At 800 MHz, radiation patterns of the ports of the MMA are shown in Fig. 2, and the matrix of the beam cosines of the radiated power is approximately given by

$$\left(\rho_{R\alpha\beta}\right) \approx \begin{pmatrix} 1.000 & 0.308 & -0.029 & 0.308 \\ 0.308 & 1.000 & 0.308 & -0.029 \\ -0.029 & 0.308 & 1.000 & 0.308 \\ 0.308 & -0.029 & 0.308 & 1.000 \end{pmatrix} \tag{19}$$

We observe that this matrix is real. This is caused by the particular configuration of the antennas, feeders and antenna tuner, and by the choice of the relative phases of $I_1, \ldots, I_m$. More precisely, let us assume that the electrical and electromagnetic properties of the antennas are invariant under the symmetries of the point group denoted by $C_{m\,v}$ in the Schönflies notation [15, ch. 2], each of these symmetries being associated with an appropriate permutation of the ports of the MAA, and of the internal connections of the MAA. The point group $C_{m\,v}$ contains m rotations of angle $2\pi p/m$ about an axis, where $p \in \{0,\ldots, m-1\}$, and m reflections in planes containing the axis. Without loss of generality, we may assume that this axis is the z-axis of the spherical coordinate system. For given α and β, the permutation associated with one of the reflection planes transforms port α into port β, and port β into port α. Without loss of generality, we may assume that this reflection plane contains the x-axis of the spherical coordinate system, so that a reflection by this plane transforms a direction (θ, φ) into a direction $(\theta, -\varphi)$. If we apply $\mathbf{I}_{0\,\alpha}$, we may use $\mathbf{h}_{0\,\alpha}(\theta, \varphi)$ and $\mathbf{h}_{0\,\alpha}(\theta, -\varphi)$ to denote the values of $\mathbf{h}_{0\alpha}$ in the directions (θ, φ) and $(\theta, -\varphi)$, respectively. If we apply $\mathbf{I}_{0\,\beta}$, we may use $\mathbf{h}_{0\,\beta}(\theta, \varphi)$ and $\mathbf{h}_{0\,\beta}(\theta, -\varphi)$ to denote the values of $\mathbf{h}_{0\beta}$ in the directions (θ, φ) and $(\theta, -\varphi)$, respectively. If $\mathbf{Y}_S$ is invariant under the associated permutation of the ports, we may conclude that $\mathbf{h}_{0\,\alpha}(\theta, \varphi) = \mathbf{h}_{0\,\beta}(\theta, -\varphi)$, and also that $\mathbf{h}_{0\,\alpha}(\theta, -\varphi) = \mathbf{h}_{0\,\beta}(\theta, \varphi)$. Thus, we have

$$\begin{aligned} &\mathbf{h}^*_{0\alpha}(\theta,\varphi)\mathbf{h}_{0\beta}(\theta,\varphi) + \mathbf{h}^*_{0\alpha}(\theta,-\varphi)\mathbf{h}_{0\beta}(\theta,-\varphi) \\ &\quad = \mathbf{h}^*_{0\alpha}(\theta,\varphi)\mathbf{h}_{0\beta}(\theta,\varphi) + \mathbf{h}^*_{0\beta}(\theta,\varphi)\mathbf{h}_{0\alpha}(\theta,\varphi) \\ &\quad = 2\,\mathrm{Re}\left(\mathbf{h}^*_{0\alpha}(\theta,\varphi)\mathbf{h}_{0\beta}(\theta,\varphi)\right) \end{aligned} \tag{20}$$

where Re denotes the real part. Thus, using (3), if $I_\alpha = I_\beta$, we obtain

$$\begin{aligned} P_{R\alpha\beta} &= \frac{\eta_0 k^2 |I_\alpha|^2}{8\pi^2} \\ &\times \int_0^\pi \int_0^\pi \mathrm{Re}\left(\mathbf{h}^*_{0\alpha}(\theta,\varphi)\mathbf{h}_{0\beta}(\theta,\varphi)\right)\sin\theta \; d\phi\, d\theta \end{aligned} \tag{21}$$

so that $P_{R\,\alpha\,\beta}$ is real. Consequently, the matrix of the beam cosines is also real.

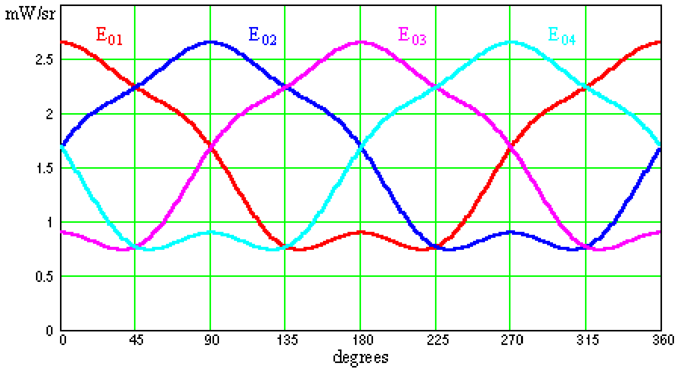

Fig. 2. Radiation patterns of the ports of the MMA, at 800 MHz, in the plane $\theta = \pi/2$, versus the azimuth angle φ, each curve corresponding to an open-circuit voltage of 2 V applied to one of the user ports of the antenna tuner.

This is what happens in [6, § 6], where the MAA is invariant under the symmetries of the point group $C_{4\,v}$ and the associated permutations. We note that the obscure explanation provided in [16] after equation (35) does not work here because the radiation patterns of the ports are not "identical patterns which are circularly symmetric", as shown in Fig. 2.

In a different configuration, the beam cosines of the radiated power need not be real.

III. Beam Cosines of the Delivered Power

Let us now consider that the MAA is used for reception. A linear multiport load (LML) has m ports numbered from 1 to m. Its admittance matrix is $\mathbf{Y}_S$. For any $\alpha \in \{1,\ldots, m\}$, port α of the MAA is coupled to port α of the LML. In a configuration where a plane wave of electric field amplitude 1 V/m rms impinges on the MAA from the direction (θ, φ), with a specified polarization which depends on (θ, φ), for any $\alpha \in \{1,\ldots, m\}$ let V_α be the rms voltage across port α of the MAA, and let $\mathbf{V}$ be the $m \times 1$ column vector of V_1 to V_m. Let g_0 be an arbitrary conductance. A plot of the power $|V_\alpha|^2/g_0$, as a function of an angle, may be referred to as a reception pattern of port α.

We assume a reciprocal MAA. We also assume a reciprocal multiport load, so that $\mathbf{Y}_S$ is symmetric. For $\alpha \in \{1,\ldots, m\}$, V_α is the open-circuit voltage of the single-port antenna defined above in § II, in the discussion of $\mathbf{E}_{0\,\alpha}$. Since $\mathbf{h}_{0\,\alpha}$ is the effective complex length of this single-port antenna in the direction (θ, φ), and using the assumed reciprocity, we have

$$V_\alpha = \mathbf{h}_{0\alpha} \cdot \mathbf{E}_{i0} \tag{22}$$

where $\mathbf{E}_{i\,0}$ is the incident electric field at the origin of the coordinate system [8, § 5.2]. Thus, if the polarization of the incident field is matched to the polarization of the single-port antenna, that is to say for $\mathbf{E}_{i\,0}$ equal to a complex constant times the complex conjugate of $\mathbf{h}_{0\,\alpha}$ [8, § 5.2] [12, § 3.3.2], we get

$$|V_\alpha| = \|\mathbf{h}_{0\alpha}\|\,\|\mathbf{E}_{i0}\| = \sqrt{\mathbf{h}_{0\alpha} \cdot \overline{\mathbf{h}_{0\alpha}}}\,\sqrt{\mathbf{E}_{i0} \cdot \overline{\mathbf{E}_{i0}}} \tag{23}$$

so that the reception pattern of port α of the MAA clearly corresponds to the directivity pattern of the single-port antenna, hence to the radiation pattern of port α of the MAA.

Let $\mathbf{P}_D$ be the matrix of the self- and cross complex powers delivered by the ports of the MAA, averaged over all directions of arrival of a plane wave of electric field amplitude 1 V/m rms impinging on the MAA with a specified polarization depending on (θ, φ), defined as follows:

$$\mathbf{P}_D = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \mathbf{V}\,\mathbf{I}^* \sin\theta \; d\varphi\, d\theta \tag{24}$$

where $\mathbf{I}$ is the $m \times 1$ column vector of the rms current flowing out of the m ports of the MAA. In (24), $\mathbf{V}$ depends on (θ, φ) according to (22), so that $\mathbf{I}$ also depends on (θ, φ). We have

$$\mathbf{P}_D = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \mathbf{V}\,\mathbf{V}^*\mathbf{Y}_S^* \sin\theta \; d\varphi\, d\theta \tag{25}$$

Let P_{del} be the average power delivered by the ports of the MAA, averaged over all directions of arrival of a plane wave of electric field amplitude 1 V/m rms impinging on the MAA with the specified polarization. We have

$$P_{\text{del}} = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \mathbf{V}^* \frac{\mathbf{Y}_S + \mathbf{Y}_S^*}{2}\mathbf{V} \sin\theta \; d\varphi\, d\theta \tag{26}$$

In the special case where $\mathbf{Y}_S = g_0\ \mathbf{1}_m$ and where $\mathbf{1}_m$ is the identity matrix of size m by m, we get

$$\mathbf{P}_D = \frac{g_0}{4\pi}\int_0^{\pi}\int_0^{2\pi} \mathbf{V}\,\mathbf{V}_0^* \sin\theta \; d\varphi\, d\theta \tag{27}$$

so that, for $\alpha \in \{1,..., m\}$ and $\beta \in \{1,..., m\}$, the entry $P_{D\alpha\beta}$ of $\mathbf{P}_D$ is given by

$$P_{D\alpha\beta} = \frac{g_0}{4\pi}\int_0^{\pi}\int_0^{2\pi} V_\alpha \overline{V_\beta} \sin\theta \; d\varphi\, d\theta \tag{28}$$

Using (22), we obtain in this special case

$$P_{D\alpha\beta} = \frac{g_0}{4\pi}\int_0^{\pi}\int_0^{2\pi} \left(\mathbf{h}_{0\alpha}\cdot\mathbf{E}_{i0}\right)\left(\overline{\mathbf{h}_{0\beta}\cdot\mathbf{E}_{i0}}\right) \sin\theta \; d\varphi\, d\theta \tag{29}$$

which is, in general, quite different from $P_{R\alpha\beta}$ given by (3). In this special case, we also have

$$P_{\text{del}} = \frac{g_0}{4\pi}\int_0^{\pi}\int_0^{2\pi} \mathbf{V}^*\mathbf{V} \sin\theta \; d\varphi\, d\theta = \operatorname{tr}\mathbf{P}_D \tag{30}$$

where $\operatorname{tr}\mathbf{P}_D$ stands for the trace of $\mathbf{P}_D$.

Since, for any $\alpha \in \{1,..., m\}$ and any $\beta \in \{1,..., m\}$, by (28), $P_{D\alpha\beta}$ is an hermitian product of the beams V_α and V_β, we can, in the special case where $\mathbf{Y}_S = g_0\ \mathbf{1}_m$, define the beam cosines of the delivered power, denoted by $\rho_{D\alpha\beta}$ and given by

$$\rho_{D\alpha\beta} = \frac{P_{D\alpha\beta}}{\sqrt{P_{D\alpha\alpha}P_{D\beta\beta}}} \tag{31}$$

The matrix $(\rho_{D\alpha\beta})$ is Hermitian. Its entries are remotely related to the correlation coefficients discussed in [16, § III].

Beam cosines of the delivered power were computed in [6, § 6] for the same MAA and the same $\mathbf{Y}_S = 20\,\text{mS} \times \mathbf{1}_4$ as in the computation of the beam cosines of the radiated power mentioned above in § II. At 800 MHz, the matrix of the beam cosines of the delivered power is approximately given by

$$\left(\rho_{D\alpha\beta}\right) \approx \begin{pmatrix} 1.000 & 0.308 & -0.029 & 0.308 \\ 0.308 & 1.000 & 0.308 & -0.029 \\ -0.029 & 0.308 & 1.000 & 0.308 \\ 0.308 & -0.029 & 0.308 & 1.000 \end{pmatrix} \tag{32}$$

We observe that the right-hand sides of (19) and (32) are equal. To explain this, we note that each antenna had the same vertical polarization, so that $\mathbf{h}_{0\alpha}$ was collinear to the unit vector $\mathbf{e}_\theta$ of the spherical coordinate, and the polarization of the incident wave was matched because $\mathbf{E}_{i0}$ was in the form $\mathbf{E}_{i0} = E_{i0}\,\mathbf{e}_\theta$, the coordinate E_{i0} being real, so that (29) became

$$P_{D\alpha\beta} = \frac{g_0\left|E_{i0}\right|^2}{4\pi}\int_0^{\pi}\int_0^{2\pi} h_{0\alpha}\overline{h_{0\beta}} \sin\theta \; d\varphi\, d\theta \tag{33}$$

where $|E_{i0}| = 1$ V/m, and where $h_{0\alpha}$ and $h_{0\beta}$ are defined by $\mathbf{h}_{0\alpha} = h_{0\alpha}\,\mathbf{e}_\theta$ and $\mathbf{h}_{0\beta} = h_{0\beta}\,\mathbf{e}_\theta$. Moreover, since $I_\alpha = I_\beta$, (3) became

$$P_{R\alpha\beta} = \frac{\eta_0 k^2\left|I_\alpha\right|^2}{16\pi^2}\int_0^{\pi}\int_0^{2\pi} \overline{h_{0\alpha}}\, h_{0\beta} \sin\theta \; d\varphi\, d\theta \tag{34}$$

where $|I_\alpha| = 40$ mA. $P_{R\alpha\beta}$ being real for the reasons explained in § II, we have $P_{R\alpha\beta} = P_{R\beta\alpha}$. As a consequence, in the particular configuration and for the assumptions used in [6, § 6], the matrix of the beam cosines of the radiated power is equal to the matrix of the beam cosines of the delivered power.

IV. Radiation Efficiency of the MAA

Some authors consider that the radiation efficiency of a port of an MAA can be defined as the ratio of the total radiated power to the maximum power available from a single port source connected to the port, the one or more other ports of the MAA being connected to a specified multiport load [17]. This is not a good choice because:

- the radiation efficiency is originally defined, for a single port antenna, as the ratio of the total radiated power to the net power accepted by the single antenna port (that is, the power received by the single antenna port, as opposed to a forward power delivered by the source connected to it) [18, p. 30];
- a configuration in which a single port source is connected to a port of an MAA, the one or more other ports of the MAA being connected to a specified multiport load, does not represent the intended use of a typical MAA.

We define the radiation efficiency of the MAA, denoted by e, as the ratio of the total radiated power to the power received by the ports of the MAA, that is

$$e = \frac{P_{\text{rad}}}{P_{\text{ANT}}} \tag{35}$$

where we use the notations of § II and assume that $P_{\text{ANT}} \neq 0$ W. We get

$$e = \frac{\mathbf{I}_0^*\,\mathbf{Z}_{\text{rad}}\,\mathbf{I}_0}{\mathbf{I}_0^*\,\mathbf{Z}_{\text{pow}}\,\mathbf{I}_0} \tag{36}$$

where we have assumed that the denominator is nonzero. In (36), e is a function of the complex nonzero vector $\mathbf{I}_0$. Thus, e

depends on the excitation. Moreover, e is real and $e \geq 0$. Power conservation entails $e \leq 1$, so that we have $0 \leq e \leq 1$.

Let $\mathbf{A}$ be a positive definite matrix. We know that [13, § 7.2] there exists a unique positive definite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. The matrix $\mathbf{B}$ is referred to as the unique positive definite square root of $\mathbf{A}$, and is denoted by $\mathbf{A}^{1/2}$. It satisfies $(\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$, and we write $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$. $\mathbf{Z}_{\text{pow}}$ is positive semidefinite as explained above in § 4.1. Assuming that $\mathbf{Z}_{\text{pow}}$ is positive definite, we can introduce the new variable $\mathbf{x} = \mathbf{Z}_{\text{pow}}{}^{1/2}\,\mathbf{I}_0$. Since $\mathbf{I}_0 = \mathbf{Z}_{\text{pow}}{}^{-1/2}\,\mathbf{x}$, we have

$$e = \left(\frac{\mathbf{x}^*\mathbf{M}\,\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right)_{\mathbf{x}\neq\mathbf{0}} \tag{37}$$

where

$$\mathbf{M} = \mathbf{Z}_{\text{pow}}^{-1/2}\,\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}^{-1/2} \tag{38}$$

The matrix $\mathbf{M}$ is clearly Hermitian. Since $e \geq 0$, $\mathbf{M}$ is positive semidefinite. Let us use $\lambda_1,..., \lambda_m$ to denote the eigenvalues of $\mathbf{M}$, counting multiplicity, which are real, these eigenvalues being labeled in ascending order. By the Rayleigh-Ritz theorem [13, § 4.2] and (37), we have

$$0 \leq \lambda_1 = \min_{\mathbf{x}\neq 0}\left(\frac{\mathbf{x}^*\mathbf{M}\,\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right) \leq e \leq \lambda_m = \max_{\mathbf{x}\neq 0}\left(\frac{\mathbf{x}^*\mathbf{M}\,\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right) \leq 1 \tag{39}$$

Consequently, we have found that, if $\mathbf{Z}_{\text{pow}}$ is positive definite, λ_1 and λ_m are the minimum value and the maximum value of e, respectively, when $\mathbf{I}_0$ takes on any possible nonzero value. At this stage, to obtain λ_1 and λ_m, we need to compute $\mathbf{M}$ using (38), and then to compute its eigenvalues. The computation can be simplified significantly if we observe that

$$\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}^{-1} = \mathbf{Z}_{\text{pow}}^{1/2}\,\mathbf{M}\,\mathbf{Z}_{\text{pow}}^{-1/2} \tag{40}$$

so that $\mathbf{M}$ is similar to $\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}{}^{-1}$. Thus, $\mathbf{M}$ and $\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}{}^{-1}$ have the same eigenvalues, counting multiplicity [13, § 1.3]. Consequently $\lambda_1,..., \lambda_m$ are the eigenvalues of $\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}{}^{-1}$, counting multiplicity, which are real, these eigenvalues being labeled in ascending order.

In the case where $\mathbf{I}_0$ is known, a value of e can be computed, which lies in $[\lambda_1, \lambda_m]$. In the case where $\mathbf{I}_0$ is not known, $\mathbf{I}_0$ can be considered as a random complex vector. In this case, if we had suitable information on the statistics of $\mathbf{I}_0$, we could derive the expectation $< e >$ of e, which lies in $[\lambda_1, \lambda_m]$. Following a different approach, we note that, according to the Courant-Fischer "min-max theorem" [13, § 4.2] each eigenvalue of $\mathbf{M}$ is a stationary value of e. We can define an "average" value of e, denoted by e_{MP}, as the average of these eigenvalues. Since

$$\sum_{i=1}^{m} \lambda_i = \operatorname{tr}\mathbf{M} = \operatorname{tr}\left(\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}^{-1}\right) \tag{41}$$

our average value e_{MP} is given by

$$e_{MP} = \frac{1}{n}\operatorname{tr}\left(\mathbf{Z}_{\text{rad}}\,\mathbf{Z}_{\text{pow}}^{-1}\right) \tag{42}$$

We note that e_{MP} lies in $[\lambda_1, \lambda_m]$, and that e_{MP} can be regarded as an expectation of e for an assumed statistics of $\mathbf{I}_0$.

V. Conclusion

We have studied the matrix of the beam cosines of the radiated power of an MAA and the matrix of the beam cosines of the delivered power of the MAA. These matrices are complex and they need not be equal. However, we have explained why, in some special cases, they can be real and equal.

We have defined the radiation efficiency of an MAA, in a manner that is consistent with the definition used for a single antenna. The radiation efficiency depends on the excitation. However, a minimum value, a maximum value and an average value of the radiation efficiency can be computed.

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