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Some Results on Power in Passive Linear Time-Invariant Multiports, Part 1

FRÉDÉRIC BROYDÉ[1], and EVELYNE CLAVELIER[2]

[1]Excem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France
[2]Eurexcem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France

Corresponding author: Frédéric Broydé (e-mail: fredbroyde@eurexcem.com).

ABSTRACT We investigate a reciprocal and passive linear time-invariant multiport, having a port set coupled to a generator and a port set coupled to a load, in the harmonic steady state. Two configurations are considered, in which the port set at which the generator is connected and the port set at which the load is connected are exchanged. We establish a new reciprocal theorem about the bounds of the set of the values of the transducer power gain obtained for all nonzero excitations, in the two configurations. For the case where the two port sets have the same number of ports, we also state and prove a new reciprocal theorem about the bounds of the set of the values of the insertion power gain obtained for all nonzero excitations, in the two configurations.

INDEX TERMS Reciprocity, transducer power gain, insertion power gain, passive circuits, linear circuits, circuit theory.

I. INTRODUCTION

This article is a revised and expanded version of the material presented in [1]–[2].

A device under study (DUS) is a linear time-invariant (LTI) and passive 2-port operating in the harmonic steady state, at a given frequency. It is used in two configurations, which are shown in Fig. 1. In configuration A (CA), its port 1 is connected to an LTI generator of internal impedance Z_{S1} and its port 2 is connected to an LTI load of impedance Z_{S2}. In configuration B (CB) its port 1 is connected to an LTI load of impedance Z_{S1} and its port 2 is connected to an LTI generator of internal impedance Z_{S2}. Let us use:

- P_{AAVG1} to denote the average power available from the generator at port 1, in CA;
- P_{ADP2} to denote the average power delivered by port 2, in CA;
- P_{BAVG2} to denote the average power available from the generator at port 2, in CB; and
- P_{BDP1} to denote the average power delivered by port 1, in CB.

To ensure that P_{AAVG1} and P_{BAVG2} are defined, we assume that the real parts of Z_{S1} and Z_{S2} are both positive. We assume that the DUS is a reciprocal device, which in this paper refers to the definitions of reciprocal networks provided in [3, Ch. 1] or [4, Ch. 16], which are not limited to lumped networks (see Appendix). Ignoring noise power

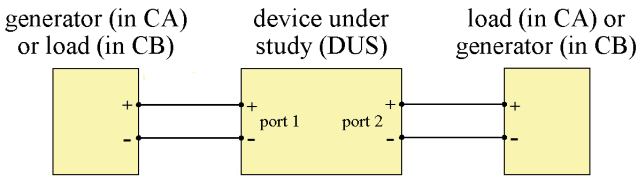

FIGURE 1. The two configurations, CA and CB, considered in the introduction.

contributions, and assuming nonzero P_{AAVG1} and P_{BAVG2}, we have

$$\frac{P_{ADP2}}{P_{AAVG1}} = \frac{P_{BDP1}}{P_{BAVG2}} \,. \tag{1}$$

This reciprocal relation means that the transducer power gains are equal in the two configurations. It was stated and proven in [5], using power waves. A less general version had been established 35 years earlier, using the entries of the impedance matrix of the DUS [6].

This paper is about power in passive LTI multiports. Our proofs are based on Section II, which introduces broadened definitions of parallel-augmented multiports and series-augmented multiports, and provides new results concerning them. In Section III, these new results are compared to known properties of augmented networks [7]–[10].

Sections IV to VI present new theoretical developments on power ratios relating to any DUS which is an LTI and passive multiport having one or more input ports and one or more output ports. The main result of Section IV is a reciprocal theorem on the transducer power gain (Theorem 4), which extends (1) to such a DUS. Section V presents a new theoretical development which, in the case where an insertion power gain may be defined, leads to a reciprocal theorem on the insertion power gain (Theorem 6). The reciprocal theorems are partially applicable to any LTI DUS, and fully applicable to a reciprocal DUS in which bidirectional signaling or power transfer takes place, such as the one presented in Section VII.

II. PRELIMINARIES

We use $\mathbf{M}^*$ to denote the hermitian adjoint of an arbitrary complex matrix $\mathbf{M}$. Recall that, if $\mathbf{M}$ is square, the hermitian part of $\mathbf{M}$, denoted by $H(\mathbf{M})$, is the matrix given by

$$H(\mathbf{M}) = \frac{\mathbf{M} + \mathbf{M}^*}{2} . \tag{2}$$

It is well-known that a positive definite matrix is invertible, and that its inverse is positive definite [11, Sec. 7.2.1]. The following lemma is more general.

Lemma 1. Let $\mathbf{M}$ be a square complex matrix. If $H(\mathbf{M})$ is positive definite, then $\mathbf{M}$ is invertible and $H(\mathbf{M}^{-1})$ is positive definite.

Proof: By the Ostrowski-Taussky determinant inequality [11, Sec. 7.8.19]–[12], if $H(\mathbf{M})$ is positive definite, then $|\det\mathbf{M}|$ is positive. Thus, $\mathbf{M}$ is invertible. The fact that $H(\mathbf{M}^{-1})$ is positive definite is for instance proven in [13], using the theory of pencils of hermitian forms, and in particular Theorem 22 of Chapter X of [14]. □

It is useful to clarify the vocabulary which will be used in what follows. We only consider the harmonic steady state, at a given frequency. Infinity is not a real number. We consider that resistance, reactance, conductance and susceptance are real (real numbers or real functions), so that they cannot be infinite. Infinity is not a complex number. We consider that impedance and admittance are complex (complex numbers or complex functions), so that they cannot be infinite. Thus, a port having zero admittance has no impedance, and a port having zero impedance has no admittance. We use $\mathrm{Re}(z)$ to denote the real part of the complex number z.

We consider an LTI multiport having N ports, where N is an integer greater than or equal to one, the N ports being numbered from 1 to N. This multiport is referred to as the "original multiport". At the given frequency, the original multiport need not have an impedance matrix, because:

- it need not be possible to inject an arbitrary current in any one of its ports (i.e. one of its ports may present a zero admittance), in a setup where its other ports are open-circuited, as for instance shown in the examples of Fig. 2 (a) and (b); and
- when it is possible to inject an arbitrary current in one of its ports, in a setup where its other ports are open-circuited, then the voltage across each of its ports need not be finite, e.g., in the 2-port shown in Fig. 2 (c), at its resonant frequency and excited at its port 1.

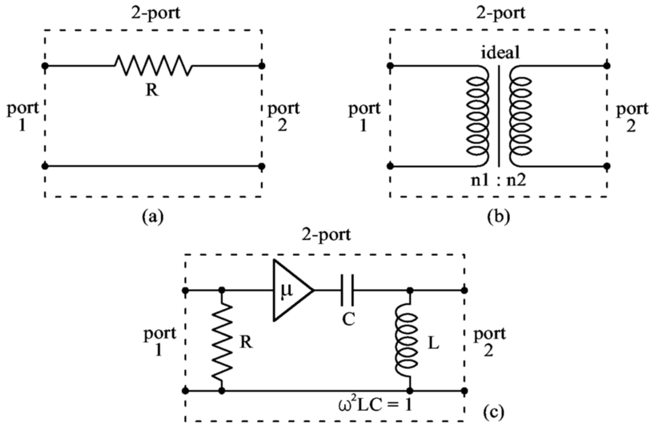

FIGURE 2. Three LTI 2-ports which do not have an impedance matrix: (a) comprises only a single resistor; (b) comprises only an ideal transformer; and (c) comprises a series resonant circuit driven, at its resonant frequency, by an ideal voltage amplifier (i.e., a dependent voltage source) of gain μ.

Likewise, at the given frequency, the original multiport need not have an admittance matrix, because:

- it need not be possible to apply an arbitrary voltage to any one of its ports (i.e. one of its ports may present a zero impedance), in a setup where its other ports are short-circuited; and
- when it is possible to apply an arbitrary voltage to one of its ports, in a setup where its other ports are short-circuited, then the current flowing into each of its ports need not be finite.

We note that, for the original multiport shown in Fig. 2 (c), a Laplace domain impedance matrix exists for $\mathrm{Re}(s) > 0$, which can be used to describe and predict the behavior of this circuit in the Laplace and time domains. However, in the harmonic steady state considered in this paper, this multiport has no impedance matrix at the resonant frequency.

According to these considerations, we need to take into account the possibility of infinite voltages or currents occurring at the ports of the original multiport.

In what follows, we assume that the original multiport is passive, and we also consider another LTI multiport, referred to as the "added multiport". The added multiport is arbitrary, but we assume that it has N ports numbered from 1 to N, and that, at any frequency, it has an impedance matrix having a positive definite hermitian part, or an admittance matrix having a positive definite hermitian part.

Lemma 2. The added multiport has the following properties:

(a) at any frequency, the added multiport has an impedance matrix, denoted by $\mathbf{Z}_A$, and an admittance matrix $\mathbf{Y}_A = \mathbf{Z}_A^{-1}$;
(b) at any frequency, the matrices $\mathbf{Z}_A$ and $\mathbf{Y}_A$ each have a positive definite hermitian part;
(c) the average power received by the added multiport, denoted by P_A, satisfies $P_A \geqslant 0$ W, in other words the added multiport is passive;

(d) we have $P_A = 0\,\text{W}$ if and only if the voltage across each port of the added multiport is 0 V, or equivalently if and only if the current flowing into each port of the added multiport is 0 A;
(e) if P_A is finite, the absolute value of the voltage across any port of the added multiport must be finite, and the absolute value of the current flowing into any port of the added multiport must be finite.

Proof: The results (a) and (b) are direct consequences of Lemma 1. For any $p \in \{1, \ldots, N\}$, let v_p be the complex rms voltage across port p. Since $\mathbf{Y}_A$ exists, for any $(v_1, \ldots, v_N)$ the average power received by the added multiport is

$$P_A = \overline{\begin{pmatrix} v_1 & \cdots & v_N \end{pmatrix}} \, \frac{\mathbf{Y}_A + \mathbf{Y}_A^*}{2} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} , \tag{3}$$

where the horizontal bar represents the complex conjugate. This may be written

$$P_A = \mathbf{V}^* H(\mathbf{Y}_A) \mathbf{V} \quad \text{where} \quad \mathbf{V} = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} . \tag{4}$$

The results (c) and (d) follow from the assumption that $H(\mathbf{Y}_A)$ is positive definite. To prove (e), we investigate the power dissipation associated to an arbitrary $\mathbf{V}$. Let λ_{min} be the smallest eigenvalue of $H(\mathbf{Y}_A)$. Using Rayleigh's theorem [11, Sec. 4.2.2], we obtain

$$\lambda_{min} \mathbf{V}^* \mathbf{V} \leqslant \mathbf{V}^* H(\mathbf{Y}_A) \mathbf{V} . \tag{5}$$

Since $H(\mathbf{Y}_A)$ is positive definite, $\lambda_{min} > 0\,\text{S}$ and we may conclude that, for any integer $q \in \{1, \ldots, N\}$,

$$|v_q|^2 \leqslant \mathbf{V}^* \mathbf{V} \leqslant \frac{P_A}{\lambda_{\min}} . \tag{6}$$

Thus, any infinite voltage would require an infinite power dissipation. Thus, if P_A is finite, the absolute value of the voltage across any port of the added multiport must be finite. A similar conclusion for currents may be obtained by utilizing $P_A = \mathbf{I}^* H(\mathbf{Z}_A) \mathbf{I}$ instead of (4). □

We can make up for the fact that the original multiport need not have an impedance matrix, in the following way. For any integer $p \in \{1, \ldots, N\}$, we can connect port p of the original multiport in parallel with port p of the added multiport, as shown in Fig. 3 for $N = 2$, to obtain a new multiport, referred to as the parallel-augmented multiport, having N ports numbered from 1 to N. The parallel-augmented multiport is LTI and it follows from Lemma 2 (c) that it is passive.

Theorem 1. At any frequency, the parallel-augmented multiport has an impedance matrix, denoted by $\mathbf{Z}_{PAM}$, which depends on $\mathbf{Y}_A$ and has a positive semidefinite hermitian part. Moreover, if the added multiport is a reciprocal device (i.e., if $\mathbf{Y}_A$ is symmetric) and the original multiport is a reciprocal device, then $\mathbf{Z}_{PAM}$ is symmetric.

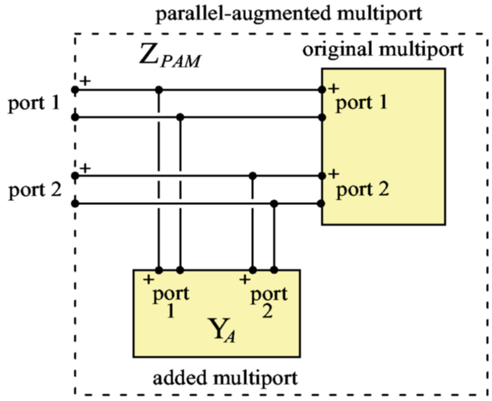

FIGURE 3. The parallel-augmented multiport, for $N = 2$.

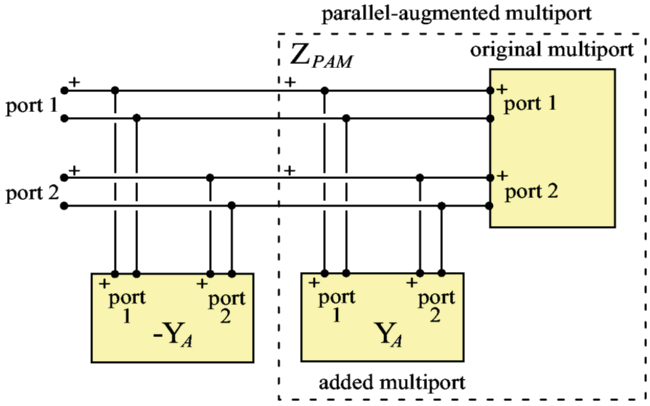

FIGURE 4. A first equivalent circuit of the original multiport, for $N = 2$.

Proof: For any $p \in \{1, \ldots, N\}$, let us consider port p of the parallel-augmented multiport, in a setup where its other ports are open-circuited.

If port p does not have an impedance of 0 Ω, we can apply a complex rms voltage $v_p = 1\,\text{V}$ at port p. Since the original multiport is passive, the parallel-augmented multiport receives a power P which satisfies $P \geqslant P_A$, where P_A is the power received by the added multiport in this configuration. Since $v_p = 1\,\text{V}$, we have $P_A > 0\,\text{W}$ by Lemma 2 (d). Thus, a complex current i_p flows into port p of the parallel-augmented multiport and $\text{Re}(i_p) > 0\,\text{A}$, because $P = v_p \text{Re}(i_p) > 0\,\text{W}$. Thus, under our assumption, port p of the parallel-augmented multiport has an impedance, having a positive real part.

If we no longer assume that port p of the parallel-augmented multiport does not have an impedance of 0 Ω, we can say that this port p has an impedance, denoted by Z, such that $\text{Re}(Z) \geqslant 0\,\Omega$. Thus, a current source delivering a complex rms current $i_p = 1\,\text{A}$ may be connected to port p of the parallel-augmented multiport. This current source produces a finite voltage $Z\, i_p$ across port p. The original multiport being passive, we have

$$P_A \leqslant |i_p|^2 \, \text{Re}(Z) . \tag{7}$$

which shows that P_A is finite. Thus, by Lemma 2 (e), we can say that, for any $q \in \{1, \ldots, N\}$, $|v_q|$ is finite, and

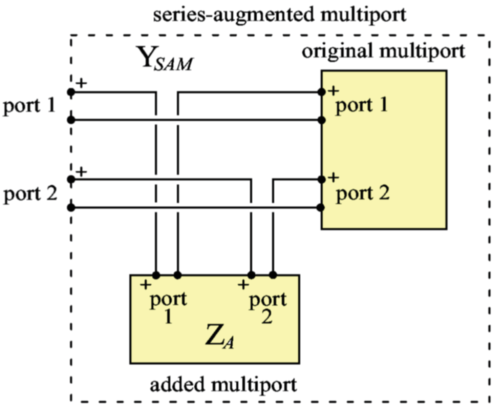

FIGURE 5. The series-augmented multiport, for $N = 2$.

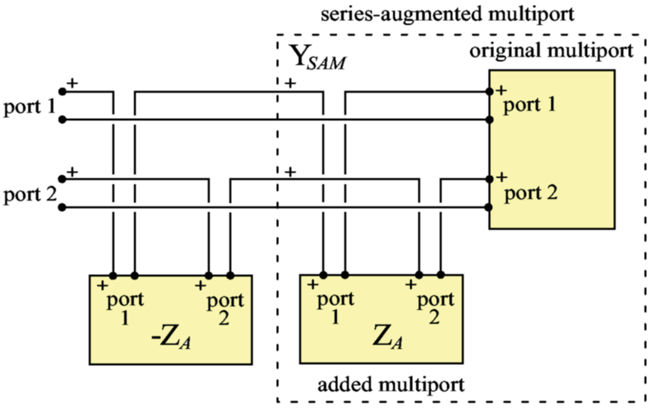

FIGURE 6. A second equivalent circuit of the original multiport, for $N = 2$.

corresponds to a transfer impedance if $q \neq p$, or to the impedance Z if $q = p$.

Since all this can be done for any $p \in \{1, \ldots, N\}$, we can determine an impedance matrix of the parallel-augmented multiport, which has a positive semidefinite hermitian part because the parallel-augmented multiport is passive. Proving the last statement of the theorem, concerning reciprocity, is simple or involved, according to the definition of reciprocity used. This statement is proven in the Appendix. □

Corollary 1. The original multiport has an equivalent circuit, shown in Fig. 4 for $N = 2$, comprising the parallel-augmented multiport and a multiport having N ports numbered from 1 to N, of admittance matrix $-\mathbf{Y}_A$, the equivalent circuit being such that, for any integer $p \in \{1, \ldots, N\}$, port p of this multiport is connected in parallel with port p of the parallel-augmented multiport. Consequently, if the original multiport has an admittance matrix $\mathbf{Y}$, then: $\mathbf{Z}_{PAM}$ is invertible; $\mathbf{Z}_{PAM}^{-1} = \mathbf{Y} + \mathbf{Y}_A$; and, if $\mathbf{Y}_A$ is symmetric, $\mathbf{Z}_{PAM}$ is symmetric if and only if $\mathbf{Y}$ is symmetric.

We can also make up for the fact that the original multiport need not have an admittance matrix, in a different way. For any $p \in \{1, \ldots, N\}$, we can connect port p of the original multiport in series with port p of the added multiport, as shown in Fig. 5 for $N = 2$, to obtain a new multiport, referred to as the series-augmented multiport, having N ports numbered from 1 to N. The series-augmented multiport is LTI and it follows from Lemma 2 (c) that it is passive.

Theorem 2. At any frequency, the series-augmented multiport has an admittance matrix, denoted by $\mathbf{Y}_{SAM}$, which depends on $\mathbf{Z}_A$ and has a positive semidefinite hermitian part. Moreover, if the added multiport is a reciprocal device (i.e., if $\mathbf{Z}_A$ is symmetric) and the original multiport is a reciprocal device, then $\mathbf{Y}_{SAM}$ is symmetric.

The proof of Theorem 2 is similar to the proof of Theorem 1 and is consequently omitted.

Corollary 2. The original multiport has an equivalent circuit, shown in Fig. 6 for $N = 2$, comprising the series-augmented multiport and a multiport having N ports numbered from one to N, of impedance matrix $-\mathbf{Z}_A$, the equivalent circuit being such that, for any integer $p \in \{1, \ldots, N\}$, port p of this multiport is connected in series with port p of the series-augmented multiport. Consequently, if the original multiport has an impedance matrix $\mathbf{Z}$, then: $\mathbf{Y}_{SAM}$ is invertible; $\mathbf{Y}_{SAM}^{-1} = \mathbf{Z} + \mathbf{Z}_A$; and, if $\mathbf{Z}_A$ is symmetric, $\mathbf{Y}_{SAM}$ is symmetric if and only if $\mathbf{Z}$ is symmetric .

III. COMPARISON TO EARLIER USES OF AUGMENTED NETWORKS

A particular series-augmented multiport (referred to as “augmented network”), in which the added multiport is made of N resistors of nonzero resistance R_0 each connected in series with one of the ports of the original multiport, was used by Carlin in [7] and Oono in [8] to define the scattering matrix of the original multiport, by

$$\mathbf{S} = \mathbf{1}_N - 2R_0 \mathbf{Y}_{SAM}\,, \tag{8}$$

where $\mathbf{S}$ is the scattering matrix of the original multiport for the reference resistance R_0, and $\mathbf{1}_N$ is the identity matrix of size N by N. These authors assumed the existence of $\mathbf{Y}_{SAM}$ as a premise, so that (8) proved that $\mathbf{S}$ always exists.

A particular parallel-augmented multiport, in which the added multiport is made of resistors of nonzero conductance G_0 each connected in parallel with one of the ports of the original multiport, is mentioned in [7], where it is said to be also suitable to define the scattering matrix of the original multiport.

Thus, the existence of $\mathbf{Y}_{SAM}$ and $\mathbf{Z}_{PAM}$ at any (real) frequency is postulated in [7] and [8], on account of the added resistors, without additional explanation. The existence of $\mathbf{Y}_{SAM}$ at any frequency is a consequence of Theorem 2 of a paper of Youla, Castriota and Carlin [9]–[10], the proof of which is involved and based on several assumptions (whose physical significance is not elementary). In section 3.3 of [15], a particular series-augmented multiport and a particular parallel-augmented multiport are introduced, in connection with the definition of scattering matrices. In both cases, the added multiport has a diagonal impedance matrix, and the existence of $\mathbf{Y}_{SAM}$ and $\mathbf{Z}_{PAM}$ is regarded as obvious.

Consequently, we may say that the series-augmented multiport and the parallel-augmented multiport defined in Section III are more general than the ones considered in [7]–[10] and [15], and that we have provided a new and simple proof of the existence of $\mathbf{Y}_{SAM}$ and $\mathbf{Z}_{PAM}$ at any frequency. The existence of $\mathbf{Z}_{PAM}$ is instrumental in what follows, but we could have used $\mathbf{Y}_{SAM}$ instead of $\mathbf{Z}_{PAM}$.

IV. THEOREMS ON THE TRANSDUCER POWER GAIN

In what follows, we use rms values for the phasors of voltages and currents, and we ignore noise power contributions.

A device under study (DUS) is a passive LTI multiport having 2 sets of ports, referred to as port set 1 and port set 2. Port set 1 consists of m ports numbered from 1 to m, and port set 2 consists of n ports numbered from 1 to n, where m and n are integers greater than or equal to 1. The DUS is an $(m+n)$-port. In what follows, when we say that port set 1 is connected to an m-port device, it is assumed that the ports of the m-port device are numbered from 1 to m, and that, for any integer $p \in \{1, \ldots, m\}$, its port p is connected to port p of port set 1 (positive terminal to positive terminal and negative terminal to negative terminal). Likewise, when we say that port set 2 is connected to an n-port device, it is assumed that the ports of the n-port device are numbered from 1 to n, and that, for any integer $q \in \{1, \ldots, n\}$, its port q is connected to port q of port set 2 (positive terminal to positive terminal and negative terminal to negative terminal).

The DUS operates in the harmonic steady state, at a given frequency f_G. It is used in two configurations, which are shown in Fig. 7. In configuration A (CA), port set 1 is connected to an LTI m-port generator of internal impedance matrix $\mathbf{Z}_{S1}$ at f_G, and port set 2 is connected to an LTI n-port load of impedance matrix $\mathbf{Z}_{S2}$ at f_G. In configuration B (CB), port set 1 is connected to an LTI m-port load of impedance matrix $\mathbf{Z}_{S1}$ at f_G, and port set 2 is connected to an LTI n-port generator of internal impedance matrix $\mathbf{Z}_{S2}$ at f_G. Let us use:

- P_{AAVG1} to denote the average power available from the generator at port set 1, in CA;
- P_{ADP2} to denote the average power delivered by port set 2, in CA;
- P_{BAVG2} to denote the average power available from the generator at port set 2, in CB; and
- P_{BDP1} to denote the average power delivered by port set 1, in CB.

We assume that the hermitian parts of $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are positive definite. By Lemma 1, we can define $\mathbf{Y}_{S1} = \mathbf{Z}_{S1}^{-1}$ and $\mathbf{Y}_{S2} = \mathbf{Z}_{S2}^{-1}$, the hermitian parts of $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ being both positive definite. It also follows from Lemma 1 that, instead of assuming that $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ exist and that $H(\mathbf{Z}_{S1})$ and $H(\mathbf{Z}_{S2})$ are positive definite, we could equivalently have assumed that $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ exist and that $H(\mathbf{Y}_{S1})$ and $H(\mathbf{Y}_{S2})$ are positive definite.

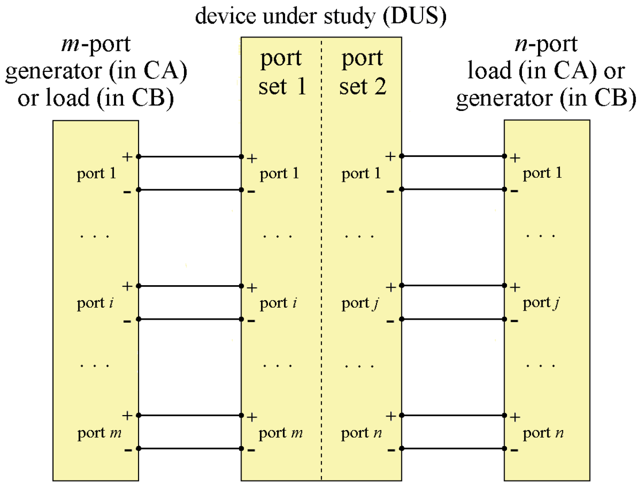

FIGURE 7. The two configurations, CA and CB, considered in Section IV and Section V.

Let $\mathbf{I}_{S1}$ and $\mathbf{V}_{O1}$ be the column vectors of the short-circuit currents and of the open-circuit voltages of the m-port generator at port set 1 in CA, respectively. $H(\mathbf{Y}_{S1})$ and $H(\mathbf{Z}_{S1})$ being positive definite, $\mathbf{Y}_{S1} + \mathbf{Y}^*_{S1}$ and $\mathbf{Z}_{S1} + \mathbf{Z}^*_{S1}$ are invertible, so that the power available from the m-port generator at port set 1 in CA is defined and given by [16]–[17]:

$$P_{AAVG1} = \frac{1}{2}\mathbf{I}^*_{S1}\left(\mathbf{Y}_{S1} + \mathbf{Y}^*_{S1}\right)^{-1}\mathbf{I}_{S1}\,, \tag{9}$$

or

$$P_{AAVG1} = \frac{1}{2}\mathbf{V}^*_{O1}\left(\mathbf{Z}_{S1} + \mathbf{Z}^*_{S1}\right)^{-1}\mathbf{V}_{O1}\,. \tag{10}$$

By [11, Sec. 7.2.1], $(\mathbf{Y}_{S1} + \mathbf{Y}^*_{S1})^{-1}$ and $(\mathbf{Z}_{S1} + \mathbf{Z}^*_{S1})^{-1}$ are positive definite. Thus, P_{AAVG1} is nonzero if and only if $\mathbf{I}_{S1}$ is nonzero, or, equivalently, if and only if $\mathbf{V}_{O1}$ is nonzero.

Let $\mathbf{I}_{S2}$ and $\mathbf{V}_{O2}$ be the column vectors of the short-circuit currents and of the open-circuit voltages of the n-port generator at port set 2 in CB, respectively. $H(\mathbf{Y}_{S2})$ and $H(\mathbf{Z}_{S2})$ being positive definite, $\mathbf{Y}_{S2} + \mathbf{Y}^*_{S2}$ and $\mathbf{Z}_{S2} + \mathbf{Z}^*_{S2}$ are invertible, so that the power available from the n-port generator at port set 2 in CB is defined and given by:

$$P_{BAVG2} = \frac{1}{2}\mathbf{I}^*_{S2}\left(\mathbf{Y}_{S2} + \mathbf{Y}^*_{S2}\right)^{-1}\mathbf{I}_{S2}\,, \tag{11}$$

or

$$P_{BAVG2} = \frac{1}{2}\mathbf{V}^*_{O2}\left(\mathbf{Z}_{S2} + \mathbf{Z}^*_{S2}\right)^{-1}\mathbf{V}_{O2}\,. \tag{12}$$

Since $(\mathbf{Y}_{S2} + \mathbf{Y}^*_{S2})^{-1}$ and $(\mathbf{Z}_{S2} + \mathbf{Z}^*_{S2})^{-1}$ are positive definite, P_{BAVG2} is nonzero if and only if $\mathbf{I}_{S2}$ is nonzero, or, equivalently, if and only if $\mathbf{V}_{O2}$ is nonzero.

At this stage, we know that the transducer power gain in CA, given by P_{ADP2}/P_{AAVG1}, is defined for any nonzero $\mathbf{V}_{O1}$ and for any nonzero $\mathbf{I}_{S1}$; and that the transducer power gain in CB, given by P_{BDP1}/P_{BAVG2}, is defined for any nonzero $\mathbf{V}_{O2}$ and for any nonzero $\mathbf{I}_{S2}$.

We consider the ports of the DUS in the following order: ports 1 to m of port set 1, and then ports 1 to n of port set 2. Let us introduce a parallel-augmented multiport composed of the DUS (as original multiport), of an m-port load of impedance matrix $\mathbf{Z}_{S1}$ connected to port set 1, and of an n-port load of impedance matrix $\mathbf{Z}_{S2}$ connected to port set 2. Here, the impedance matrix of the added multiport is

$$\mathbf{Z}_A = \begin{pmatrix} \mathbf{Z}_{S1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_{S2} \end{pmatrix} . \tag{13}$$

The hermitian parts of $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ being positive definite, it follows that the hermitian part of $\mathbf{Z}_A$ is positive definite. By Theorem 1, the parallel-augmented multiport has an impedance matrix $\mathbf{Z}_{PAM}$. The matrix $\mathbf{Z}_{PAM}$ is of size $(m+n)$ by $(m+n)$ and it may be partitioned into four submatrices, $\mathbf{Z}_{PAM11}$ of size m by m, $\mathbf{Z}_{PAM12}$ of size m by n, $\mathbf{Z}_{PAM21}$ of size n by m and $\mathbf{Z}_{PAM22}$ of size n by n, which are such that

$$\mathbf{Z}_{PAM} = \begin{pmatrix} \mathbf{Z}_{PAM11} & \mathbf{Z}_{PAM12} \\ \mathbf{Z}_{PAM21} & \mathbf{Z}_{PAM22} \end{pmatrix} . \tag{14}$$

Using Corollary 1, by inspection, we obtain

$$P_{ADP2} = \mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* \frac{\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*}{2} \mathbf{Z}_{PAM21} \mathbf{I}_{S1} \tag{15}$$

and

$$P_{BDP1} = \mathbf{I}_{S2}^* \mathbf{Z}_{PAM12}^* \frac{\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*}{2} \mathbf{Z}_{PAM12} \mathbf{I}_{S2} . \tag{16}$$

Using (9), (11), (15) and (16), we find that the transducer power gains in CA and CB are given by

$$\frac{P_{ADP2}}{P_{AAVG1}} = \frac{\mathbf{I}_{S1}^* \mathbf{Z}_{PAM21}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM21} \mathbf{I}_{S1}}{\mathbf{I}_{S1}^* (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1} \mathbf{I}_{S1}} \tag{17}$$

and

$$\frac{P_{BDP1}}{P_{BAVG2}} = \frac{\mathbf{I}_{S2}^* \mathbf{Z}_{PAM12}^* (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*) \mathbf{Z}_{PAM12} \mathbf{I}_{S2}}{\mathbf{I}_{S2}^* (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{-1} \mathbf{I}_{S2}} , \tag{18}$$

respectively. Since these ratios depend on $\mathbf{I}_{S1}$ and $\mathbf{I}_{S2}$, (1) cannot apply here, except in very special cases. Consequently, some work is needed to generalize (1) to the DUS considered here.

Let $\mathbf{A}$ be a positive definite matrix. We know that there exists a unique positive definite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$ [11, Sec. 7.2.6]. The matrix $\mathbf{B}$ is referred to as the unique positive definite square root of $\mathbf{A}$, and is denoted by $\mathbf{A}^{1/2}$. It satisfies $(\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$. This allows us to write $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$. Since $H(\mathbf{Y}_{S1})$ and $H(\mathbf{Y}_{S2})$ are positive definite, we can define the matrices

$$\mathbf{M}_1 = (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{1/2} \, \mathbf{Z}_{PAM21}^* \times (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*) \mathbf{Z}_{PAM21} (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{1/2} , \tag{19}$$

which is of size m by m, and

$$\mathbf{M}_2 = (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{1/2} \, \mathbf{Z}_{PAM12}^* \times (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*) \mathbf{Z}_{PAM12} (\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{1/2} , \tag{20}$$

which is of size n by n. $\mathbf{M}_1$ and $\mathbf{M}_2$ are clearly hermitian, so that their eigenvalues are real. Note that the eigenvalues of $\mathbf{M}_1$ and $\mathbf{M}_2$ are dimensionless numbers, since $\mathbf{M}_1$ and $\mathbf{M}_2$ are dimensionless matrices.

Theorem 3. The matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ defined by (19) and (20) are positive semidefinite, so that their eigenvalues are nonnegative. Let $\lambda_{1\max}$ be the largest eigenvalue of $\mathbf{M}_1$ and $\lambda_{1\min}$ the smallest eigenvalue of $\mathbf{M}_1$. Let $\lambda_{2\max}$ be the largest eigenvalue of $\mathbf{M}_2$ and $\lambda_{2\min}$ the smallest eigenvalue of $\mathbf{M}_2$. We have

$$0 \leqslant \lambda_{1\min} \leqslant \lambda_{1\max} \leqslant 1 , \tag{21}$$

$$0 \leqslant \lambda_{2\min} \leqslant \lambda_{2\max} \leqslant 1 , \tag{22}$$

$$0 \leqslant \lambda_{1\min} P_{AAVG1} \leqslant P_{ADP2} \leqslant \lambda_{1\max} P_{AAVG1} , \tag{23}$$

and

$$0 \leqslant \lambda_{2\min} P_{BAVG2} \leqslant P_{BDP1} \leqslant \lambda_{2\max} P_{BAVG2} . \tag{24}$$

Moreover,

- the equality $P_{ADP2} = \lambda_{1\max} P_{AAVG1}$ is satisfied if $\mathbf{I}_{S1}$ is the product of $(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{1/2}$ and an eigenvector of $\mathbf{M}_1$ associated with $\lambda_{1\max}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$;
- the equality $P_{ADP2} = \lambda_{1\min} P_{AAVG1}$ is satisfied if $\mathbf{I}_{S1}$ is the product of $(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{1/2}$ and an eigenvector of $\mathbf{M}_1$ associated with $\lambda_{1\min}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$;
- the equality $P_{BDP1} = \lambda_{2\max} P_{BAVG2}$ is satisfied if $\mathbf{I}_{S2}$ is the product of $(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{1/2}$ and an eigenvector of $\mathbf{M}_2$ associated with $\lambda_{2\max}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$; and
- the equality $P_{BDP1} = \lambda_{2\min} P_{BAVG2}$ is satisfied if $\mathbf{I}_{S2}$ is the product of $(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{1/2}$ and an eigenvector of $\mathbf{M}_2$ associated with $\lambda_{2\min}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$.

Moreover, if $\mathbf{Z}_{PAM}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric, we have:

- $\lambda_{1\max} = \lambda_{2\max}$;
- if $m = n$, then $\lambda_{1\min} = \lambda_{2\min}$;
- if $m > n$, then $\lambda_{1\min} = 0$; and
- if $m < n$, then $\lambda_{2\min} = 0$.

Proof: The hermitian part of $\mathbf{Y}_{S2}$ being positive definite, $\mathbf{M}_1$ is positive semidefinite by [11, Sec. 7.1.8], so that its eigenvalues are nonnegative by [11, Sec. 7.1.4]. For CA, let us introduce the new variable $\mathbf{X}_1 = (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1/2} \, \mathbf{I}_{S1}$. Since $\mathbf{I}_{S1} = (\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{1/2} \, \mathbf{X}_1$, it follows from (9), (15) and (19) that

$$P_{AAVG1} = \frac{1}{2} \mathbf{X}_1^* \mathbf{X}_1 \quad \text{and} \quad P_{ADP2} = \frac{1}{2} \mathbf{X}_1^* \mathbf{M}_1 \mathbf{X}_1 . \tag{25}$$

By Rayleigh's theorem [11, Sec. 4.2.2], we have

$$0 \leqslant \lambda_{1\min} \mathbf{X}_1^* \mathbf{X}_1 \leqslant \mathbf{X}_1^* \mathbf{M}_1 \mathbf{X}_1 \leqslant \lambda_{1\max} \mathbf{X}_1^* \mathbf{X}_1 , \tag{26}$$

which, used with (25), proves (23). The other assertions of Theorem 3 relating to $\mathbf{M}_1$ also result from Rayleigh's theorem and the definition of $\mathbf{X}_1$. The fact that $\lambda_{1\max} \leqslant 1$ is a consequence of the fact that there exists a value of $\mathbf{X}_1$ for which $P_{ADP2} = \lambda_{1\max} P_{AAVG1}$, while the passivity of the DUS entails $P_{ADP2} \leqslant P_{AAVG1}$. The arguments for the assertions of Theorem 3 relating to $\mathbf{M}_2$ and for $\lambda_{2\max} \leqslant 1$ are similar.

Since $(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)^{1/2}$ and $(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)^{1/2}$ are invertible square matrices, it follows from [11, Sec. 1.3.22] that $\mathbf{M}_1$ has the same eigenvalues, counting multiplicity, as

$$\mathbf{N}_1 = \mathbf{Z}_{PAM21}^* \times (\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)\,, \quad (27)$$

which is of size m by m; and that $\mathbf{M}_2$ has the same eigenvalues, counting multiplicity, as

$$\mathbf{N}_2 = \mathbf{Z}_{PAM12}^* \times (\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM12}(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)\,, \quad (28)$$

which is of size n by n. If $\mathbf{Z}_{PAM}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric, then $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ are symmetric and the transpose of $\mathbf{Z}_{PAM12}$ is $\mathbf{Z}_{PAM21}$, so that the transpose of $\mathbf{N}_2$ is

$$\mathbf{N}_2^T = (\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*) \times \mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM21}^*\,. \quad (29)$$

By [11, Sec. 1.4.1], the eigenvalues of $\mathbf{N}_2^T$ are the same as those of $\mathbf{M}_2$, counting multiplicity. We can then observe that the right hand sides of (27) and (29) are $\mathbf{Z}_{PAM21}^*\mathbf{B}$ and $\mathbf{B}\,\mathbf{Z}_{PAM21}^*$, respectively, where $\mathbf{B}$ is the matrix given by $(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)$. Consequently, using [11, Sec. 1.3.22] again and the fact that $\mathbf{Z}_{PAM21}^*$ is of size m by n, we find that:

- if $m = n$, then $\mathbf{M}_1$ and $\mathbf{M}_2$ have the same eigenvalues, counting multiplicity;
- if $m > n$, then $\mathbf{M}_1$ has the same eigenvalues as $\mathbf{M}_2$, counting multiplicity, together with $m-n$ additional eigenvalues equal to zero; and
- if $m < n$, then $\mathbf{M}_2$ has the same eigenvalues as $\mathbf{M}_1$, counting multiplicity, together with $n-m$ additional eigenvalues equal to zero.

This leads to the final assertion of Theorem 3. □

Observation 1. We note that, if we only need the eigenvalues of $\mathbf{M}_1$ or $\mathbf{M}_2$, the shortest computation is a direct computation of the eigenvalues of $\mathbf{N}_1$ or $\mathbf{N}_2$ defined by (27) and (28).

Using Theorem 3, we get the new *Reciprocal theorem on the transducer power gain,* which reads as follows.

Theorem 4. Ignoring noise power contributions and using the notations of Theorem 3, we can assert that:

(a) the set of the values of the transducer power gain in CA, that is of $G_{TA} = P_{ADP2}/P_{AAVG1}$, obtained for all nonzero $\mathbf{V}_{O1}$, or equivalently for all nonzero $\mathbf{I}_{S1}$, has a least element referred to as "minimum value", equal to $\lambda_{1\min}$, and a greatest element referred to as "maximum value", equal to $\lambda_{1\max}$;

(b) the set of the values of the transducer power gain in CB, that is of $G_{TB} = P_{BDP1}/P_{BAVG2}$, obtained for all nonzero $\mathbf{V}_{O2}$, or equivalently for all nonzero $\mathbf{I}_{S2}$, has a least element referred to as "minimum value", equal to $\lambda_{2\min}$, and a greatest element referred to as "maximum value", equal to $\lambda_{2\max}$;

(c) if the DUS and both loads are reciprocal devices, the maximum value of G_{TA} and the maximum value of G_{TB} are equal to $\lambda_{1\max} = \lambda_{2\max}$; and

(d) if the DUS and both loads are reciprocal devices, and if $m = n$, then the minimum value of G_{TA} and the minimum value of G_{TB} are equal to $\lambda_{1\min} = \lambda_{2\min}$.

V. THEOREMS ON THE INSERTION POWER GAIN

For $n = m$, let P_{AW} be the power which would be received by the n-port load connected at port set 2 in CA, if the DUS was not present and this n-port load was directly connected to the m-port generator connected at port set 1 in CA. We note that $H(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})$ is positive definite, so that, by Lemma 1, $\mathbf{Y}_{S1}+\mathbf{Y}_{S2}$ is invertible. We have

$$P_{AW} = \mathbf{I}_{S1}^*(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1*} \times \frac{\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*}{2}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1}\mathbf{I}_{S1}\,, \quad (30)$$

so that the insertion power gain of the DUS in CA is given by

$$\frac{P_{ADP2}}{P_{AW}} = \frac{\mathbf{I}_{S1}^*\mathbf{Z}_{PAM21}^*(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21}\mathbf{I}_{S1}}{[\mathbf{I}_{S1}^*(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1*}(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*) \times(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1}\mathbf{I}_{S1}]}\,, \quad (31)$$

where we have used (15). Let P_{BW} be the power which would be received by the m-port load connected at port set 1 in CB, if the DUS was not present and this m-port load was directly connected to the n-port generator connected at port set 2 in CB. Using again the fact that $\mathbf{Y}_{S1}+\mathbf{Y}_{S2}$ is invertible, we find

$$P_{BW} = \mathbf{I}_{S2}^*(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1*} \times \frac{\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*}{2}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1}\mathbf{I}_{S2}\,, \quad (32)$$

so that the insertion power gain of the DUS in CB is given by

$$\frac{P_{BDP1}}{P_{BW}} = \frac{\mathbf{I}_{S2}^*\mathbf{Z}_{PAM12}^*(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM12}\mathbf{I}_{S2}}{[\mathbf{I}_{S2}^*(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1*}(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*) \times(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1}\mathbf{I}_{S2}]}\,, \quad (33)$$

where we have used (16).

The hermitian parts of $\mathbf{Y}_{S1}$ and $\mathbf{Y}_{S2}$ being positive definite, we may conclude that the matrices

$$\mathbf{L}_1 = (\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1*}(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1} \quad (34)$$

and

$$\mathbf{L}_2 = (\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1*}(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^{-1} \quad (35)$$

are hermitian and positive definite. Thus, we can define the matrices

$$\mathbf{M}_1 = \mathbf{L}_1^{-1/2}\mathbf{Z}_{PAM21}^*(\mathbf{Y}_{S2}+\mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21}\mathbf{L}_1^{-1/2} \quad (36)$$

and

$$\mathbf{M}_2 = \mathbf{L}_2^{-1/2}\mathbf{Z}_{PAM12}^*(\mathbf{Y}_{S1}+\mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM12}\mathbf{L}_2^{-1/2} \quad (37)$$

Since $\mathbf{L}_1^{-1/2}$ and $\mathbf{L}_2^{-1/2}$ are hermitian, $\mathbf{M}_1$ and $\mathbf{M}_2$ are hermitian, so that their eigenvalues are real. Note that the eigenvalues of $\mathbf{M}_1$ and $\mathbf{M}_2$ are dimensionless numbers, since $\mathbf{M}_1$ and $\mathbf{M}_2$ are dimensionless matrices.

Theorem 5. The matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ defined by (36) and (37) are positive semidefinite, so that their eigenvalues are nonnegative. Let $\lambda_{1\max}$ be the largest eigenvalue of $\mathbf{M}_1$ and $\lambda_{1\min}$ the smallest eigenvalue of $\mathbf{M}_1$. Let $\lambda_{2\max}$ be the largest eigenvalue of $\mathbf{M}_2$ and $\lambda_{2\min}$ the smallest eigenvalue of $\mathbf{M}_2$. We have

$$0 \leqslant \lambda_{1\min} \leqslant \lambda_{1\max}\,, \tag{38}$$

$$0 \leqslant \lambda_{2\min} \leqslant \lambda_{2\max}\,, \tag{39}$$

$$0 \leqslant \lambda_{1\min} P_{AW} \leqslant P_{ADP2} \leqslant \lambda_{1\max} P_{AW}\,, \tag{40}$$

and

$$0 \leqslant \lambda_{2\min} P_{BW} \leqslant P_{BDP1} \leqslant \lambda_{2\max} P_{BW}\,. \tag{41}$$

Moreover,

- the equality $P_{ADP2} = \lambda_{1\max} P_{AW}$ is satisfied if $\mathbf{I}_{S1}$ is the product of $\mathbf{L}_1^{-1/2}$ and an eigenvector of $\mathbf{M}_1$ associated with $\lambda_{1\max}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$;
- the equality $P_{ADP2} = \lambda_{1\min} P_{AW}$ is satisfied if $\mathbf{I}_{S1}$ is the product of $\mathbf{L}_1^{-1/2}$ and an eigenvector of $\mathbf{M}_1$ associated with $\lambda_{1\min}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$;
- the equality $P_{BDP1} = \lambda_{2\max} P_{BW}$ is satisfied if $\mathbf{I}_{S2}$ is the product of $\mathbf{L}_2^{-1/2}$ and an eigenvector of $\mathbf{M}_2$ associated with $\lambda_{2\max}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$; and
- the equality $P_{BDP1} = \lambda_{2\min} P_{BW}$ is satisfied if $\mathbf{I}_{S2}$ is the product of $\mathbf{L}_2^{-1/2}$ and an eigenvector of $\mathbf{M}_2$ associated with $\lambda_{2\min}$, measured in $\mathrm{A}^{1/2}\mathrm{V}^{1/2}$.

Moreover, if $\mathbf{Z}_{PAM}$ is symmetric and if there exist two complex numbers Z_{S1} and Z_{S2} such that $\mathbf{Z}_{S1} = Z_{S1}\mathbf{1}_m$ and $\mathbf{Z}_{S2} = Z_{S2}\mathbf{1}_n$, then $\lambda_{1\max} = \lambda_{2\max}$ and $\lambda_{1\min} = \lambda_{2\min}$.

Moreover, if $\mathbf{Z}_{PAM}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric and if $\mathbf{Z}_{PAM21}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, then $\lambda_{1\max} = \lambda_{2\max}$ and $\lambda_{1\min} = \lambda_{2\min}$.

Proof: The hermitian part of $\mathbf{Y}_{S2}$ being positive definite, $\mathbf{M}_1$ is positive semidefinite by [11, Sec. 7.1.8], so that its eigenvalues are nonnegative by [11, Sec. 7.1.4]. For CA, let us introduce the new variable $\mathbf{X}_1 = \mathbf{L}_1^{1/2}\mathbf{I}_{S1}$. Since $\mathbf{I}_{S1} = \mathbf{L}_1^{-1/2}\mathbf{X}_1$, it follows from (15), (30), (34) and (36) that

$$P_{AW} = \frac{1}{2}\mathbf{X}_1^*\mathbf{X}_1 \quad\text{and}\quad P_{ADP2} = \frac{1}{2}\mathbf{X}_1^*\mathbf{M}_1\mathbf{X}_1\,. \tag{42}$$

By Rayleigh's theorem, we have

$$0 \leqslant \lambda_{1\min}\mathbf{X}_1^*\mathbf{X}_1 \leqslant \mathbf{X}_1^*\mathbf{M}_1\mathbf{X}_1 \leqslant \lambda_{1\max}\mathbf{X}_1^*\mathbf{X}_1\,, \tag{43}$$

which, used with (42), proves (40). The other assertions of Theorem 5 relating to $\mathbf{M}_1$ also result from Rayleigh's theorem and the definition of $\mathbf{X}_1$. The arguments for the assertions of Theorem 5 relating to $\mathbf{M}_2$ are similar.

It follows from [11, Sec. 1.3.22] that $\mathbf{M}_1$ has the same eigenvalues, counting multiplicity, as

$$\mathbf{N}_1 = \mathbf{Z}_{PAM21}^*(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21}\mathbf{L}_1^{-1}\,, \tag{44}$$

and that $\mathbf{M}_2$ has the same eigenvalues, counting multiplicity, as

$$\mathbf{N}_2 = \mathbf{Z}_{PAM12}^*(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM12}\mathbf{L}_2^{-1}\,. \tag{45}$$

Using (34)-(35) in (44)-(45), we get

$$\mathbf{N}_1 = \mathbf{Z}_{PAM21}^*(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)\mathbf{Z}_{PAM21} \times (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{-1}(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*\,, \tag{46}$$

and

$$\mathbf{N}_2 = \mathbf{Z}_{PAM12}^*(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM12} \times (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1}(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*\,. \tag{47}$$

If $\mathbf{Z}_{PAM}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are symmetric, the transpose of $\mathbf{Z}_{PAM12}$ is $\mathbf{Z}_{PAM21}$ so that the transpose of $\mathbf{N}_2$ is

$$\mathbf{N}_2^T = (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1} \times (\mathbf{Y}_{S1} + \mathbf{Y}_{S2})\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)\mathbf{Z}_{PAM21}^*\,. \tag{48}$$

We need an additional assumption, suitable to allow us to remove: $(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)$ and $(\mathbf{Y}_{S2} + \mathbf{Y}_{S2}^*)^{-1}$ from (46); and $(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)$ and $(\mathbf{Y}_{S1} + \mathbf{Y}_{S1}^*)^{-1}$ from (48). A first possibility is that we assume that there exist two complex numbers Z_{S1} and Z_{S2} such that $\mathbf{Z}_{S1} = Z_{S1}\mathbf{1}_m$ and $\mathbf{Z}_{S2} = Z_{S2}\mathbf{1}_n$. A second possibility is that we assume that $\mathbf{Z}_{PAM21}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, because circulant matrices commute, linear combinations of circulant matrices are circulant, and the inverse of an invertible circulant matrix is circulant [11, Sec. 0.9.6]. Using either assumption, we obtain

$$\mathbf{N}_1 = \mathbf{Z}_{PAM21}^*\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^*\,, \tag{49}$$

and

$$\mathbf{N}_2^T = \mathbf{Z}_{PAM21}(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})(\mathbf{Y}_{S1}+\mathbf{Y}_{S2})^*\mathbf{Z}_{PAM21}^*\,. \tag{50}$$

By [11, Sec. 1.4.1], the eigenvalues of $\mathbf{N}_2^T$ are the same as those of $\mathbf{M}_2$, counting multiplicity. We can then observe that the right hand sides of (49) and (50) are $\mathbf{Z}_{PAM21}^*\mathbf{B}$ and $\mathbf{B}\,\mathbf{Z}_{PAM21}^*$, respectively, where $\mathbf{B}$ is the matrix given by $\mathbf{Z}_{PAM21}(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})(\mathbf{Y}_{S1} + \mathbf{Y}_{S2})^*$. Consequently, using [11, Sec. 1.3.22] again, we find that $\mathbf{M}_1$ and $\mathbf{M}_2$ have the same eigenvalues, counting multiplicity, which directly leads to the final assertions of Theorem 5. □

Observation 2. We note that, if we only need the eigenvalues of $\mathbf{M}_1$ or $\mathbf{M}_2$, the shortest path is a direct computation of the eigenvalues of $\mathbf{N}_1$ or $\mathbf{N}_2$ given by (44) and (45).

Observation 3. If $\mathbf{Z}_{PAM}$ is symmetric, then $\mathbf{Z}_{PAM12}$ is circulant if and only if $\mathbf{Z}_{PAM21}$ is circulant.

Proof: If $\mathbf{Z}_{PAM}$ is symmetric, then $\mathbf{Z}_{PAM12}^T = \mathbf{Z}_{PAM21}$. The transpose of a circulant matrix being circulant [18], $\mathbf{Z}_{PAM12}$ is circulant if and only if $\mathbf{Z}_{PAM21}$ is circulant. □

Observation 4. If $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, if the DUS has an impedance matrix $\mathbf{Z}$ which is a 2-by-2 block matrix, the blocks of which are of size n by n and circulant, and if $\mathbf{Z}$ is invertible, then $\mathbf{Z}_{PAM}$ is a 2-by-2 block matrix, the blocks of which are of size n by n and circulant. It follows that $\mathbf{Z}_{PAM21}$ is circulant.

Proof: We can use the formula for the inverse of a 2-by-2 block matrix [11, Sec. 0.7.3], the facts that linear combinations and products of circulant matrices are circulant, and the fact that the inverse of an invertible circulant matrix is circulant [11, Sec. 0.9.6], to show that the inverse of a 2-by-2 block matrix, the blocks of which are circulant, is a 2-by-2 block matrix, the blocks of which are circulant. We can use this result thrice and Corollary 1 to get the wanted result. □

It follows from (30) that the insertion power gain in CA, given by P_{ADP2}/P_{AW}, is defined for any nonzero $\mathbf{V}_{O1}$, and for any nonzero $\mathbf{I}_{S1}$. It follows from (32), that the insertion power gain in CB, given by P_{BDP1}/P_{BW}, is defined for any nonzero $\mathbf{V}_{O2}$, and for any nonzero $\mathbf{I}_{S2}$. Thus, using Theorem 5, we obtain the new *Reciprocal theorem on the insertion power gain,* which reads as follows.

Theorem 6. If $n = m$, ignoring noise power contributions and using the notations of Theorem 5, we can assert that:

(a) the set of the values of the insertion power gain in CA, that is of $G_{IA} = P_{ADP2}/P_{AW}$, obtained for all nonzero $\mathbf{V}_{O1}$, or equivalently for all nonzero $\mathbf{I}_{S1}$, has a least element referred to as "minimum value", equal to $\lambda_{1\min}$, and a greatest element referred to as "maximum value", equal to $\lambda_{1\max}$;

(b) the set of the values of the insertion power gain in CB, that is of $G_{IB} = P_{BDP1}/P_{BW}$, obtained for all nonzero $\mathbf{V}_{O2}$, or equivalently for all nonzero $\mathbf{I}_{S2}$, has a least element referred to as "minimum value", equal to $\lambda_{2\min}$, and a greatest element referred to as "maximum value", equal to $\lambda_{2\max}$;

(c) assuming that the DUS and both loads are reciprocal devices, if there exist two complex numbers Z_{S1} and Z_{S2} such that $\mathbf{Z}_{S1} = Z_{S1}\mathbf{1}_m$ and $\mathbf{Z}_{S2} = Z_{S2}\mathbf{1}_n$, or if $\mathbf{Z}_{PAM21}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, then: the maximum value of G_{IA} and the maximum value of G_{IB} are equal to $\lambda_{1\max} = \lambda_{2\max}$; and the minimum value of G_{IA} and the minimum value of G_{IB} are equal to $\lambda_{1\min} = \lambda_{2\min}$.

VI. ADDITIONAL INVESTIGATIONS

A. USE OF AN EXTREMUM-SEEKING ALGORITHM

An extremum-seeking algorithm can be used to approximate the maximum and minimum values defined in (a) and (b) of Theorem 4 and Theorem 6, instead of computing them as eigenvalues according to Theorem 3 and Theorem 5.

Let $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^*\mathbf{x}}$ be the euclidian vector norm of an arbitrary complex column vector $\mathbf{x}$. For an arbitrary positive integer N, we use $\mathbb{S}_N$ to denote the hypersphere of the unit vectors of $\mathbb{C}^N$. It follows from (17) and (31) that the transducer power gain G_{TA} and the insertion power gain G_{IA} are not modified if $\mathbf{I}_{S1}$ is multiplied by an arbitrary complex number. Thus, to approximate the maximum and minimum values of G_{TA} and G_{IA}, an extremum-seeking algorithm may posit $\mathbf{I}_{S1} \in \mathbb{S}_m$, and further assume that one of the entries of $\mathbf{I}_{S1}$ is real and nonnegative. Likewise, it follows from (18) and (33) that, to approximate the maximum and minimum values of G_{TB} and G_{IB}, an extremum-seeking algorithm may posit $\mathbf{I}_{S2} \in \mathbb{S}_n$, and further assume that one of the entries of $\mathbf{I}_{S2}$ is real and nonnegative. These observations lead to convenient and simple parametrizations. For instance, for $m = n = 2$, the numerical algorithm can use

$$\mathbf{I}_{S1} = \begin{pmatrix} \sin\theta_1 \exp j\phi_1 \\ \cos\theta_1 \end{pmatrix} \tag{51}$$

in CA, where $\theta_1 \in [0, \pi/2]$ and $\phi_1 \in [-\pi, \pi]$, and

$$\mathbf{I}_{S2} = \begin{pmatrix} \sin\theta_2 \exp j\phi_2 \\ \cos\theta_2 \end{pmatrix} \tag{52}$$

in CB, where $\theta_2 \in [0, \pi/2]$ and $\phi_2 \in [-\pi, \pi]$. Thus, for $m = n = 2$, to estimate each maximum or minimum value defined in (a) and (b) of Theorem 4 and Theorem 6, an extremum-seeking algorithm may solve a problem having only 2 real unknowns each lying in a bounded interval.

B. FIRST EXAMPLE

In a first example, we assume that

$$\mathbf{Z}_{S1} = \begin{pmatrix} 51+39j & 19+79j \\ 27+56j & 37+61j \end{pmatrix} \Omega\,, \tag{53}$$

$$\mathbf{Z}_{S2} = \begin{pmatrix} 32+87j & 11+41j \\ 23+37j & 73+13j \end{pmatrix} \Omega\,, \tag{54}$$

and that the DUS has an impedance matrix given by

$$\mathbf{Z} = \begin{pmatrix} 89+25j & 31+11j & 31+5j & 17+40j \\ 21+3j & 59+35j & 3+62j & 40+17j \\ 3+21j & 41+29j & 73+41j & 21+49j \\ 33+13j & 7+7j & 23+42j & 49+21j \end{pmatrix} \Omega\,. \tag{55}$$

$\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are not symmetric and have each a positive definite hermitian part. Corollary 1 can be used to obtain $\mathbf{Z}_{PAM}$. The maximum and minimum values defined in (a) and (b) of Theorem 4 and Theorem 6 have been computed as eigenvalues according to Theorem 3 and Theorem 5, and independently determined by an extremum-seeking algorithm using (51) or (52). Both methods give exactly the same values, shown in Table 1.

TABLE 1. Results for the first example.

Quantity	CA	CB
maximum value of the transducer power gain	0.084966	0.171115
minimum value of the transducer power gain	0.013600	0.029740
maximum value of the insertion power gain	0.126970	0.291078
minimum value of the insertion power gain	0.048907	0.093953

Thus, if $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are not symmetric, we find that: the transducer power gain equalities stated in (c) and (d) of Theorem 4 need not be true; and the insertion power gain equalities stated in (c) of Theorem 6 need not be true.

C. SECOND EXAMPLE

In a second example, we assume that

$$\mathbf{Z}_{S1} = \begin{pmatrix} 51+39j & 23+68j \\ 23+68j & 37+61j \end{pmatrix} \Omega\,, \tag{56}$$

$$\mathbf{Z}_{S2} = \begin{pmatrix} 32+87j & 17+39j \\ 17+39j & 73+13j \end{pmatrix} \Omega\,, \tag{57}$$

and that the DUS has an impedance matrix given by

$$\mathbf{Z} = \begin{pmatrix} 89+25j & 26+7j & 17+13j & 25+27j \\ 26+7j & 59+35j & 22+46j & 24+12j \\ 17+13j & 22+46j & 73+41j & 22+46j \\ 25+27j & 24+12j & 22+46j & 49+21j \end{pmatrix} \Omega\,. \tag{58}$$

Here, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are symmetric and have each a positive definite hermitian part. Neither $\mathbf{Z}_{S1}$ nor $\mathbf{Z}_{S2}$ is in the form of a complex number times an identity matrix. Also, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are not circulant. The maximum and minimum values defined in (a) and (b) of Theorem 4 and Theorem 6 have been computed as eigenvalues according to Theorem 3 and Theorem 5, and independently determined by an extremum-seeking algorithm using (51) or (52). Both methods give exactly the same values, shown in Table 2.

TABLE 2. Results for the second example.

Quantity	CA	CB
maximum value of the transducer power gain	0.065234	0.065234
minimum value of the transducer power gain	0.018019	0.018019
maximum value of the insertion power gain	0.159534	0.141010
minimum value of the insertion power gain	0.037131	0.042008

Thus, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ being symmetric, we find that: the transducer power gain equalities stated in (c) and (d) of Theorem 4 are compatible with the computed values; and the insertion power gain equalities stated in (c) of Theorem 6 need not be true in a case where we cannot say that $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are each in the form of a complex number times an identity matrix, and where we cannot say that $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}_{PAM21}$ are circulant.

D. THIRD EXAMPLE

In a third example, we assume that

$$\mathbf{Z}_{S1} = (51+39j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Omega\,, \tag{59}$$

$$\mathbf{Z}_{S2} = (32+87j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Omega\,, \tag{60}$$

and that the DUS has an impedance matrix given by (58).

Here, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are symmetric and have each a positive definite hermitian part. Also, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are each in the form of a complex number times an identity matrix. The maximum and minimum values defined in (a) and (b) of Theorem 4 and Theorem 6 have been computed as eigenvalues according to Theorem 3 and Theorem 5, and independently determined by an extremum-seeking algorithm using (51) or (52). Both methods give exactly the same values, shown in Table 3.

TABLE 3. Results for the third example.

Quantity	CA	CB
maximum value of the transducer power gain	0.049441	0.049441
minimum value of the transducer power gain	0.017073	0.017073
maximum value of the insertion power gain	0.172413	0.172413
minimum value of the insertion power gain	0.059538	0.059538

$\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ being symmetric, we find that: the transducer power gain equalities stated in (c) and (d) of Theorem 4 are compatible with the computed values; and, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ being each in the form of a complex number times an identity matrix, the insertion power gain equalities stated in (c) of Theorem 6 are compatible with the computed values.

E. FOURTH EXAMPLE

In a fourth example, we assume that

$$\mathbf{Z}_{S1} = \begin{pmatrix} 51-39j & 7+16j \\ 7+16j & 51-39j \end{pmatrix} \Omega\,, \tag{61}$$

$$\mathbf{Z}_{S2} = \begin{pmatrix} 32+47j & 11+41j \\ 11+41j & 32+47j \end{pmatrix} \Omega\,, \tag{62}$$

and that the DUS has an impedance matrix given by

$$\mathbf{Z} = \begin{pmatrix} 54+25j & 6+7j & 20+13j & -10-5j \\ 6+7j & 54+25j & -10-5j & 20+13j \\ 20+13j & -10-5j & 25-25j & 6+17j \\ -10-5j & 20+13j & 6+17j & 25-25j \end{pmatrix} \Omega\,. \tag{63}$$

Here, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ are symmetric and have each a positive definite hermitian part. $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant, and $\mathbf{Z}$ is a 2-by-2 block matrix, the blocks of which are of size 2 by 2 and circulant. It follows from Observation 4 that $\mathbf{Z}_{PAM21}$ is circulant.

The maximum and minimum values defined in (a) and (b) of Theorem 4 and Theorem 6 have been computed as eigenvalues according to Theorem 3 and Theorem 5, and independently determined by an extremum-seeking algorithm using (51) or (52). Both methods give exactly the same values, shown in Table 4.

TABLE 4. Results for the fourth example.

Quantity	CA	CB
maximum value of the transducer power gain	0.120251	0.120251
minimum value of the transducer power gain	0.010066	0.010066
maximum value of the insertion power gain	0.215581	0.215581
minimum value of the insertion power gain	0.014556	0.014556

$\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}$ being symmetric, we find that: the transducer power gain equalities stated in (c) and (d) of Theorem 4 are compatible with the computed values; and, $\mathbf{Z}_{S1}$, $\mathbf{Z}_{S2}$ and $\mathbf{Z}_{PAM21}$ being circulant, the insertion power gain equalities stated in (c) of Theorem 6 are compatible with the computed values.

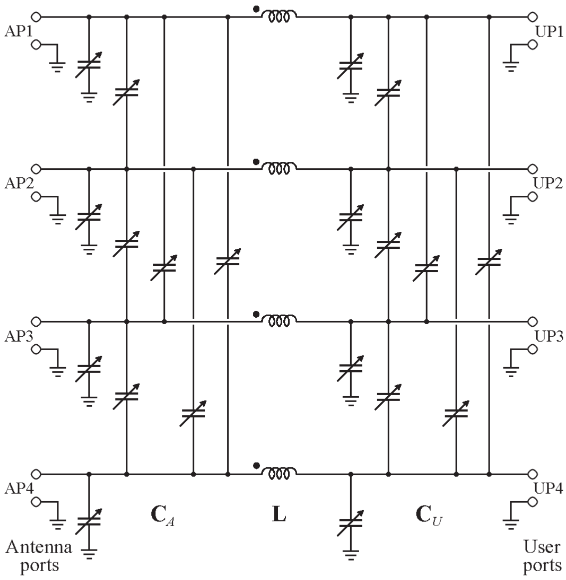

FIGURE 8. A MIMO matching circuit having the structure of a multidimensional π-network. It has $n = 4$ antenna ports, labeled AP1 to AP4, and $m = 4$ user ports, labeled UP1 to UP4.

VII. APPLICATION TO A MIMO MATCHING CIRCUIT

The reciprocal and passive LTI DUS shown in Fig. 8 is a multiple-input-port and multiple-output-port (MIMO) matching circuit in which port set 1 is composed of $m = 4$ user ports intended to be coupled to a radio transceiver, and port set 2 is composed of $n = 4$ antenna ports, each of which is intended to be connected to an antenna. This matching circuit having the structure of a multidimensional π-network has already been investigated [19]–[23]. It comprises 20 adjustable impedance devices presenting a negative reactance, each depicted using a variable capacitor symbol in Fig. 8. It can be adjusted to modify the impedance matrix presented by port set 1, denoted by $\mathbf{Z}_U$.

We assume that the radio transceiver is such that

$$\mathbf{Z}_{S1} = r_0 \mathbf{1}_4 \,, \tag{64}$$

where $r_0 = 50\,\Omega$. We note that CA corresponds to emission, and CB to reception.

The antennas are $n = 4$ side-by-side parallel dipole antennas, each having a total length of 224.8 mm. The radius of the array is 56.2 mm. Each antenna is lossless and has a 60 mm long lossy feeder. The antenna array is intended to operate in the frequency band 700 MHz to 900 MHz. At the center frequency $f_c = 800\,\text{MHz}$, $\mathbf{Z}_{S2}$ is approximately given by

$$\mathbf{Z}_{S2} = \begin{pmatrix} 8.6-8.9j & 3.8+4.9j & 1.7+2.2j & 3.8+4.9j \\ 3.8+4.9j & 8.6-8.9j & 3.8+4.9j & 1.7+2.2j \\ 1.7+2.2j & 3.8+4.9j & 8.6-8.9j & 3.8+4.9j \\ 3.8+4.9j & 1.7+2.2j & 3.8+4.9j & 8.6-8.9j \end{pmatrix} \Omega \,. \tag{65}$$

At any frequency, $\mathbf{Z}_{S2}$ is symmetric and circulant, as shown in (65) at f_c, so that $\mathbf{Z}_{S2}$ is fully determined by the first three entries of its first row. These entries are plotted in the frequency range 700 MHz to 900 MHz, in Fig. 9.

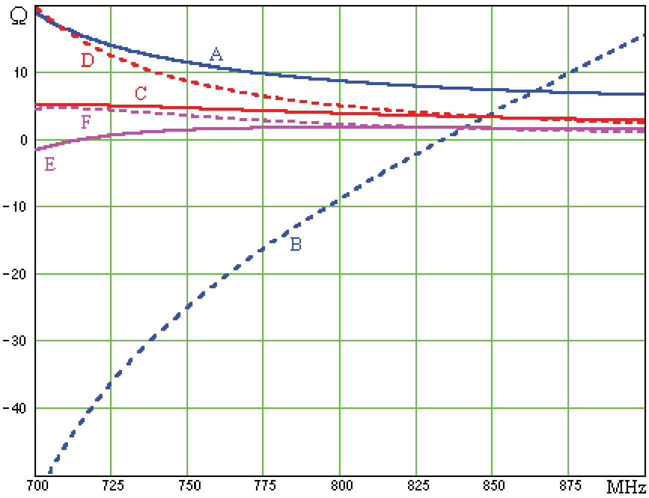

FIGURE 9. Entries of $\mathbf{Z}_{S2}$ versus frequency: $\mathrm{Re}(\mathbf{Z}_{S2\,11})$ is curve A; $\mathrm{Im}(\mathbf{Z}_{S2\,11})$ is curve B; $\mathrm{Re}(\mathbf{Z}_{S2\,12})$ is curve C; $\mathrm{Im}(\mathbf{Z}_{S2\,12})$ is curve D; $\mathrm{Re}(\mathbf{Z}_{S2\,13})$ is curve E; and $\mathrm{Im}(\mathbf{Z}_{S2\,13})$ is curve F.

At any tuning frequency f_T in this frequency range, the MIMO matching circuit is intended to be such that it can be adjusted to obtain that $\mathbf{Z}_U$ approximates a wanted impedance matrix $\mathbf{Z}_{UW}$, given by

$$\mathbf{Z}_{UW} = r_0 \mathbf{1}_4 \,. \tag{66}$$

We assume that the components of the MIMO matching circuit have the loss characteristics defined in [23, Sec. 5], in which it is shown that an adjustment such that $\mathbf{Z}_U = \mathbf{Z}_{UW}$ exists at any tuning frequency in the frequency range 700 MHz to 900 MHz, and in which the corresponding capacitance values of the adjustable impedance devices are computed.

In Fig. 10 and Fig. 11, we show results relating to the transducer power gain at the tuning frequency, as a function of the tuning frequency, in CA and CB, respectively. These results are the maximum transducer power gain with respect to the possible excitations, the minimum transducer power gain with respect to the possible excitations, and the transducer power gain for an excitation defined by

$$\mathbf{I}_{S1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{A in CA, or } \mathbf{I}_{S2} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{A in CB.} \tag{67}$$

The MAX and MIN curves were obtained using Theorem 3 and Observation 1, by computing the eigenvalues of $\mathbf{N}_1$ given by (27) in the case of Fig. 10, and the eigenvalues of $\mathbf{N}_2$ given by (28) in the case of Fig. 11. We observe that, in line with Theorem 4, the maximum transducer power gains in CA and CB are equal (the absolute value of the relative difference of the computed values is less than 10^{-14}), and the minimum transducer power gains in CA and CB are equal (the absolute value of the relative difference of the computed values is also less than 10^{-14}). For an arbitrary excitation, the

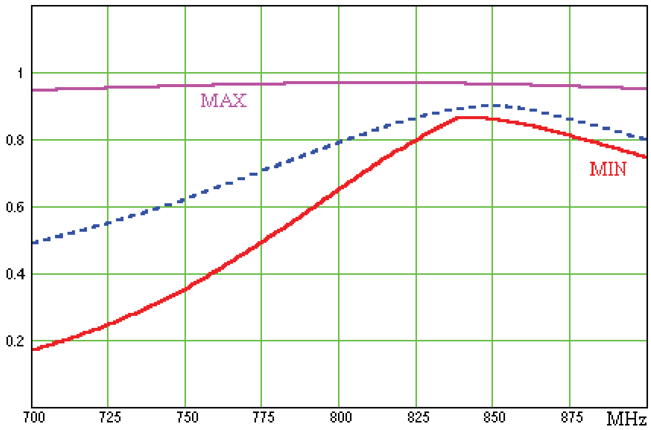

FIGURE 10. Transducer power gain at the tuning frequency in CA: the maximum value is labeled “MAX”, the minimum value is labeled “MIN”; the dashed curve corresponds to the excitation given by (67).

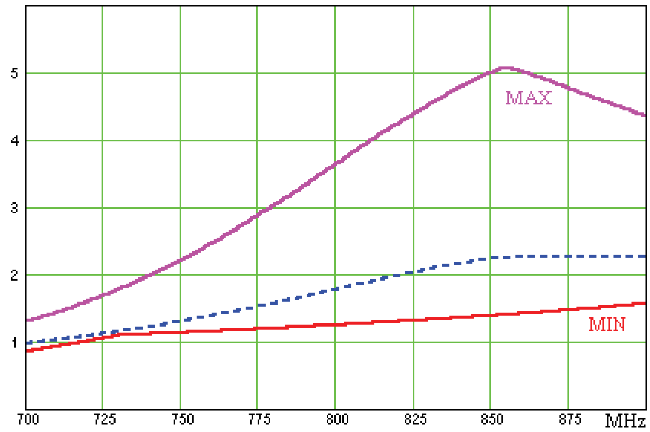

FIGURE 12. Insertion power gain at the tuning frequency in CA: the maximum value is labeled “MAX”, the minimum value is labeled “MIN”; the dashed curve corresponds to the excitation given by (67).

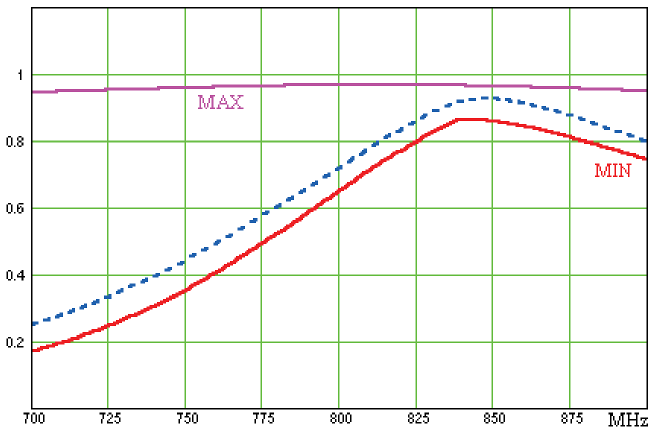

FIGURE 11. Transducer power gain at the tuning frequency in CB: the maximum value is labeled “MAX”, the minimum value is labeled “MIN”; the dashed curve corresponds to the excitation given by (67).

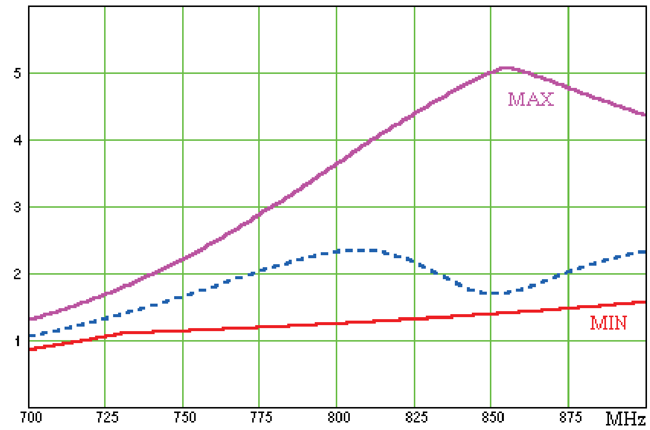

FIGURE 13. Insertion power gain at the tuning frequency in CB: the maximum value is labeled “MAX”, the minimum value is labeled “MIN”; the dashed curve corresponds to the excitation given by (67).

transducer power gain may lie anywhere between the MAX and MIN curves of Fig. 10 and Fig. 11.

In Fig. 12 and Fig. 13, we show results relating to the insertion power gain at the tuning frequency, as a function of the tuning frequency, in CA and CB, respectively. These results are the maximum insertion power gain with respect to the possible excitations, the minimum insertion power gain with respect to the possible excitations, and the insertion power gain for an excitation defined by (67).

Here, the MAX and MIN curves were obtained using Theorem 5 and Observation 2, by computing the eigenvalues of $\mathbf{N}_1$ given by (44) in the case of Fig. 12, and the eigenvalues of $\mathbf{N}_2$ given by (45) in the case of Fig. 13. We observe that the maximum insertion power gains in CA and CB are equal (the absolute value of the relative difference of the computed values is less than 10^{-12}), and the minimum insertion power gains in CA and CB are equal (the absolute value of the relative difference of the computed values is less than 10^{-14}). This is explained by Theorem 6, because the symmetry of the problem is such that $\mathbf{Z}_{PAM21}$, $\mathbf{Z}_{S1}$ and $\mathbf{Z}_{S2}$ are circulant.

We observe that the MAX and MIN curves plotted in Fig. 10 to Fig. 13 are continuous, in line with [11, Sec. 6.3.3]. They also look differentiable except at some frequencies. In fact these curves need not be differentiable at a frequency where the eigenvalues of $\mathbf{M}_1$ or $\mathbf{M}_2$, as applicable, are not distinct (see [11, Sec. 6.3.12] and [11, Sec. 6.3.P10]).

It is possible to design an adaptive MIMO antenna tuning system (also referred to as “automatic antenna tuner”), which automatically adjusts the MIMO matching circuit considered above during emission, to obtain that $\mathbf{Z}_U$ is close to $\mathbf{Z}_{UW}$ [24]–[27]. In this context, for time-division duplex (TDD) which uses the same frequency for emission and reception, we can say that Fig. 10 and Fig. 12 relate to the performance of the MIMO matching circuit during emission, versus the operating frequency; and that Fig. 11 and Fig. 13 relate to the performance of the MIMO matching circuit during reception, versus the operating frequency. Consequently, each MAX and MIN curve in these figures may be regarded as a performance criterion. As regards these criteria, our results show that the performances are the same for emission and reception. Thus, an optimal adjustment for emission, provided by the adaptive MIMO antenna tuning system, is also an optimal adjustment for reception, for these criteria. This is important from a practical standpoint.

VIII. CONCLUSION

The reciprocal theorems presented in this article are partially applicable to any passive LTI DUS. They are fully applicable and relevant to a passive and reciprocal DUS in which bidirectional signaling or power transfer takes place, such as a subcircuit of the front-end of a MIMO radio transceiver like the one studied in Section VII, a parallel multichannel electrical link (interconnect), or a system comprising two antenna arrays used to create a MIMO channel.

The reciprocal theorems provide fundamental equalities between the extrema of the transducer power gain in CA and CB, and between the extrema of the insertion power gain in CA and CB when the stated conditions are met. These theorems use a broad definition of reciprocity, which does not assume that a reciprocal device is made of lumped circuit elements. The reciprocal theorem on the transducer power gain (Theorem 4) is far more general than the reciprocal theorem on the insertion power gain (Theorem 6). It is therefore interesting to look at the differences in the proofs, which cause this important difference.

To establish the reciprocal theorems, we have used a suitable parallel-augmented multiport of the DUS, for which the impedance matrix exists and leads to simple formulas for the transducer power gain and the insertion power gain in CA and CB. However, using the same added multiport, we could also have used a series-augmented multiport, for which the admittance matrix exists and also leads to simple formulas for the transducer power gain and the insertion power gain in CA and CB.

APPENDIX

In the framework of the theory of lumped LTI circuits, a reciprocal circuit is sometimes defined as a circuit which is exclusively composed of one or more resistors, inductors, coupled inductors, capacitors and transformers, because such a circuit satisfies the reciprocity theorem [4, Ch. 16]. Based on this definition, a circuit which is exclusively composed of reciprocal circuits is obviously a reciprocal circuit.

In this paper, “reciprocity” refers to a more general definition of a reciprocal device, which is limited neither to lumped networks nor to passive networks, and only assumes that the device satisfies the conclusion of the reciprocity theorem. According to this definition, a reciprocal device is: LTI, singled-valued and such that, in the Laplace domain, all transfer admittances, transfer impedances, transfer current ratios and transfer voltage ratios corresponding to admissible signal pairs satisfy the relations stated in the conclusion of the reciprocity theorem [3, Ch. 2]–[4, Ch. 16]. Based on this definition, it is not at all obvious that a network which is exclusively composed of reciprocal devices should be a reciprocal device. This is why we need to prove the last statement of Theorem 1, according to which, if the added multiport is a reciprocal device (i.e., if $\mathbf{Y}_A$ is symmetric) and the original multiport is a reciprocal device, then $\mathbf{Z}_{PAM}$ is symmetric.

Proof: Let us first observe that, if the original multiport has an admittance matrix $\mathbf{Y}$, this matrix is symmetric, so that

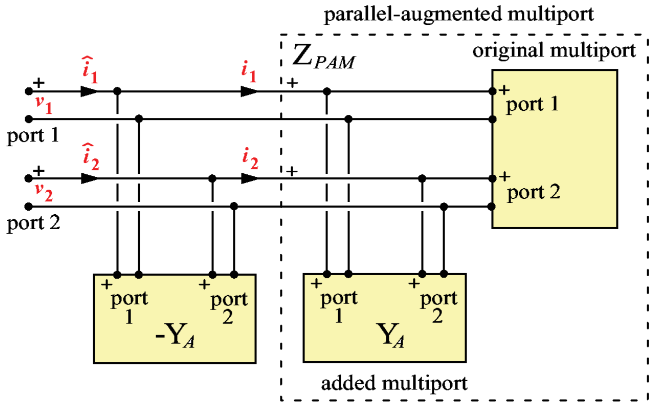

FIGURE 14. Equivalent circuit of the original multiport, for $N = 2$.

$\mathbf{Y} + \mathbf{Y}_A$ is also symmetric. Thus, $\mathbf{Z}_{PAM} = (\mathbf{Y} + \mathbf{Y}_A)^{-1}$ is also symmetric. Here, we have obtained the wanted result without much effort.

In what follows, we do not assume that the original multiport has an admittance matrix, and we consider Laplace domain voltages, currents and matrices, which depend on the Laplace variable s. As shown in Fig. 14 for $N = 2$, let $v_1, \ldots, v_N$ be the voltages at the ports of the parallel-augmented multiport, which are also the voltages at the ports of the original multiport, and $i_1, \ldots, i_N$ be the currents flowing in the ports of the parallel-augmented multiport, using associated reference directions. Let $\hat{i}_1, \ldots, \hat{i}_N$ be the currents flowing in the ports of the original multiport, using associated reference directions. By inspection, we find that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \mathbf{Z}_{PAM} \begin{pmatrix} i_1 \\ \vdots \\ i_N \end{pmatrix}, \tag{68}$$

and

$$\begin{pmatrix} \hat{i}_1 \\ \vdots \\ \hat{i}_N \end{pmatrix} + \mathbf{Y}_A \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} i_1 \\ \vdots \\ i_N \end{pmatrix}. \tag{69}$$

In (68) and (69), the vector $(i_1, \ldots, i_N)^T$ can be any complex vector of size N by 1, because $\mathbf{Z}_{PAM}$ exists. In contrast, the vectors $(v_1, \ldots, v_N)^T$ and $(\hat{i}_1, \ldots, \hat{i}_N)^T$ may be constrained to lie in a subspace of the vector space of the complex vectors of size N by 1. The original multiport being assumed to be a non-pathological and singled-valued, it is possible to select N independent variables among $v_1, \ldots, v_N, \hat{i}_1, \ldots, \hat{i}_N$, these independent variables having different indices. The original multiport creates a mapping from these independent variables to the other variables. Without loss of generality, we may assume that there exists a nonnegative integer k such that the independent variables are the entries $v_1, \ldots, v_k, \hat{i}_{k+1}, \ldots, \hat{i}_N$ of a vector $\mathbf{C}$, so that the other variables are the entries $\hat{i}_1, \ldots, \hat{i}_k, v_{k+1}, \ldots, v_N$ of a vector $\mathbf{D}$. Said mapping representing a passive LTI system, there exists a complex matrix $\mathbf{M}$ of size N by N, which is analytic in the region $\mathrm{Re}(s) > 0$ and such that [3, Ch. 2]:

$$\mathbf{D} = \mathbf{MC}. \tag{70}$$

An entry of $\mathbf{M}$ is an admittance, a transfer admittance, an impedance, a transfer impedance, a transfer current ratio or a transfer voltage ratio corresponding to admissible signal pairs. The original network satisfying the conclusion of the reciprocity theorem, it follows that an entry $M_{p\,q}$ of $\mathbf{M}$ is such that: if p and q are both lying in $\{1,\ldots,k\}$, or both lying in $\{k+1,\ldots,N\}$, we have $M_{p\,q} = M_{q\,p}$; and if $p \in \{1,\ldots,k\}$ and $q \in \{k+1,\ldots,N\}$, or if $q \in \{1,\ldots,k\}$ and $p \in \{k+1,\ldots,N\}$, then we have $M_{p\,q} = -M_{q\,p}$.

We now consider two excitations of the parallel-augmented multiport. We use the superscript a to indicate the variables corresponding to excitation a, and the superscript b to indicate the variables corresponding to excitation b. We may write

$$\sum_{p=0}^{N} v_p^a i_p^b = \sum_{p=0}^{N} (v_p^a (i_p^b - \hat{i}_p^b)) + \sum_{p=0}^{N} (v_p^a \hat{i}_p^b)\,, \qquad (71)$$

so that we obtain

$$\sum_{p=0}^{N} v_p^a i_p^b = \sum_{p=0}^{N} (v_p^a (i_p^b - \hat{i}_p^b)) + \sum_{p=0}^{k} (c_p^a d_p^b) + \sum_{p=k+1}^{N} (d_p^a c_p^b)\,. \qquad (72)$$

Introducing the entries $Y_{A p\,q}$ of $\mathbf{Y}_A$ in (69), and the entries of $\mathbf{M}$, we get

$$\sum_{p=0}^{N} v_p^a i_p^b = \sum_{p=0}^{N} \left(v_p^a \sum_{q=0}^{N} Y_{A p\,q} v_q^b \right) + \sum_{p=0}^{k} \left(c_p^a \sum_{q=0}^{N} M_{p\,q} c_q^b \right) + \sum_{p=k+1}^{N} \left(c_p^b \sum_{q=0}^{N} M_{p\,q} c_q^a \right)\,. \qquad (73)$$

Using $M_{p\,q} = -M_{q\,p}$ where it occurs, we obtain

$$\sum_{p=0}^{N} v_p^a i_p^b = \sum_{p=0}^{N} \sum_{q=0}^{N} (Y_{A p\,q} v_p^a v_q^b) + \sum_{p=0}^{k} \sum_{q=0}^{k} (M_{p\,q} c_p^a c_q^b) + \sum_{p=k+1}^{N} \sum_{q=k+1}^{N} (M_{p\,q} c_p^b c_q^a)\,, \qquad (74)$$

because

$$\sum_{p=0}^{k} \sum_{q=k+1}^{N} (M_{p\,q} c_p^a c_q^b) + \sum_{p=k+1}^{N} \sum_{q=0}^{k} (M_{p\,q} c_p^b c_q^a) = 0\,. \qquad (75)$$

We can also write

$$\sum_{p=0}^{N} v_p^b i_p^a = \sum_{p=0}^{N} (v_p^b (i_p^a - \hat{i}_p^a)) + \sum_{p=0}^{N} (v_p^b \hat{i}_p^a)\,, \qquad (76)$$

and obtain

$$\sum_{p=0}^{N} v_p^b i_p^a = \sum_{p=0}^{N} \sum_{q=0}^{N} (Y_{A p\,q} v_p^b v_q^a) + \sum_{p=0}^{k} \sum_{q=0}^{k} (M_{p\,q} c_p^b c_q^a) + \sum_{p=k+1}^{N} \sum_{q=k+1}^{N} (M_{p\,q} c_p^a c_q^b)\,. \qquad (77)$$

Using the symmetry of $\mathbf{Y}_A$ and $M_{p\,q} = M_{q\,p}$ where it occurs in (74) and (77), we get

$$\sum_{p=0}^{N} v_p^a i_p^b = \sum_{p=0}^{N} v_p^b i_p^a\,. \qquad (78)$$

Introducing the entries $Z_{PAM\,p\,q}$ of $\mathbf{Z}_{PAM}$ in (78), we get

$$\sum_{p=0}^{N} \left(\sum_{q=0}^{N} Z_{PAM\,p\,q} i_q^a \right) i_p^b = \sum_{p=0}^{N} \left(\sum_{q=0}^{N} Z_{PAM\,p\,q} i_q^b \right) i_p^a\,, \qquad (79)$$

so that

$$\sum_{p=0}^{N} \sum_{q=0}^{N} Z_{PAM\,p\,q} i_q^a i_p^b = \sum_{p=0}^{N} \sum_{q=0}^{N} Z_{PAM\,p\,q} i_p^a i_q^b\,. \qquad (80)$$

Exchanging the indices in the right-hand side of (80), we get

$$\sum_{p=0}^{N} \sum_{q=0}^{N} Z_{PAM\,p\,q} i_q^a i_p^b = \sum_{p=0}^{N} \sum_{q=0}^{N} Z_{PAM\,q\,p} i_q^a i_p^b\,, \qquad (81)$$

so that

$$\sum_{p=0}^{N} \sum_{q=0}^{N} (Z_{PAM\,p\,q} - Z_{PAM\,q\,p})\, i_q^a i_p^b = 0\,. \qquad (82)$$

Since the vector $(i_1,\ldots,i_N)^T$ can be any complex vector of size N by 1, it follows that (82) applies to any $i_1^a,\ldots,i_N^a$ and any $i_1^b,\ldots,i_N^b$. Consequently, for any $P \in \{1,\ldots,N\}$ and any $Q \in \{1,\ldots,N\}$, we can choose $i_P^b = 1$ A, we can choose $i_p^b = 0$ A for any $p \in \{1,\ldots,N\}$ such that $p \neq P$, we can choose $i_Q^a = 1$ A, and we can choose $i_q^a = 0$ A for any $q \in \{1,\ldots,N\}$ such that $q \neq Q$, so that (82) becomes $Z_{PAM\,P\,Q} = Z_{PAM\,Q\,P}$. The symmetry of $\mathbf{Z}_{PAM}$ in the region $\mathrm{Re}(s) > 0$ follows.

Since, for any $p \in \{1,\ldots,N\}$ and any $q \in \{1,\ldots,N\}$, we have $Z_{PAM\,p\,q} = Z_{PAM\,q\,p}$ for $\mathrm{Re}(s) > 0$, it follows from the uniqueness theorem on the Laplace transform [28, Sec. 8.3] that $Z_{PAM\,p\,q}$ and $Z_{PAM\,q\,p}$ are Laplace transforms of the same Laplace transformable time domain distribution. Since, by Theorem 1, $Z_{PAM\,p\,q}$ and $Z_{PAM\,q\,p}$ exist on the imaginary axis $s = j\omega$, we may conclude that they are equal on this axis. Thus, $\mathbf{Z}_{PAM}$ is symmetric on the imaginary axis $s = j\omega$. □

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FRÉDÉRIC BROYDÉ was born in France in 1960. He received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble (ENSIEG) and the Ph.D. in microwaves and microtechnologies from the Université des Sciences et Technologies de Lille (USTL).

He co-founded the Excem corporation in May 1988, a company providing engineering and research and development services. He is president and CTO of Excem. Most of his activity is allocated to engineering and research in electronics, radio, antennas, electromagnetic compatibility (EMC) and signal integrity. Currently, his most active research areas are automatic antenna tuners, measuring antennas and active antennas.

Dr. Broydé is author or co-author of about 100 technical papers, and inventor or co-inventor of about 80 patent families, for which 48 US patents have been granted. He is a Senior Member of the IEEE since 2001. He is a licensed radio amateur (F5OYE).

EVELYNE CLAVELIER was born in France in 1961. She received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble (ENSIEG).

She is co-founder of the Excem corporation, based in Maule, France. She is CEO of Excem. She is also President of Eurexcem (a subsidiary of Excem) and President of Tekcem, a company selling or licensing intellectual property rights to foster research. She is also an active engineer and researcher. Her current research area is radio communications. She has also done research work in the areas of electromagnetic compatibility (EMC) and signal integrity. She has taken part in many electronic design and software design projects of Excem.

Prior to starting Excem in 1988, she worked for Schneider Electrics (in Grenoble, France), STMicroelectronics (in Grenoble, France), and Signetics (in Mountain View, USA).

Ms. Clavelier is the author or a co-author of about 80 technical papers. She is co-inventor of about 80 patent families. She is a Senior Member of the IEEE since 2002. She is a licensed radio amateur (F1PHQ).

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