Excem Research Papers in Electronics and Electromagnetics

Number 13 June 2026 ISBN 978-2-909056-33-3

On the Current Distribution in, and the Impedance of, a Circular Loop Antenna

FRÉDÉRIC BROYDÉ[1]**, and**
EVELYNE CLAVELIER[2]

[1]Eurexcem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France
[2]Excem, 12 chemin des Hauts de Clairefontaine, 78580 Maule, France

Corresponding author: Frédéric Broydé (e-mail: fredbroyde@eurexcem.com).

ABSTRACT We investigate the current distribution in, and the impedance of, a single-turn circular wire loop antenna lying in a homogeneous and lossless medium, in the framework of a theory developed by Wu and King *et al*. Using a detailed analysis of the Wu-King factors, we explain: how the current distribution proposed by Wu for the delta-gap source can be computed accurately; and how the number of terms and the terminal-zone network needed to obtain a good estimate of the antenna impedance can be determined.

INDEX TERMS Antenna theory, loop antenna, measuring antenna, electromagnetic compatibility, EMC.

I. INTRODUCTION

This article is about a single-turn circular wire loop antenna of radius a lying in a homogeneous and lossless medium of intrinsic impedance η, permeability μ and permittivity ϵ, in which the velocity of light is c. In a previous article [1], the authors have used known explicit mathematical models of the current distribution in the loop antenna and the impedance presented by the loop antenna, to obtain mathematical models for emission and reception by the loop antenna. These models were claimed to be applicable up to the frequency

$$f_{\max} = \frac{2.5c}{2\pi a}\,. \tag{1}$$

These models use Wu-King factors denoted by A_n, where $n \in \mathbb{N}$, and [1]–[2] provide accurate and efficient formulas for the computation of the Wu-King factors.

Let ψ be the angle shown in Fig. 1. According to Wu and King *et al* [3]–[4], if the loop antenna is made of a perfect electric conductor (PEC), and is excited by a delta-gap source of voltage V_0 located at $\psi = 0$, the current distribution is accurately given by

$$i_{\mathrm{W}}(\psi) = \frac{V_0}{j\pi\eta}\left\{\frac{1}{A_0} + 2\sum_{n=1}^{\infty}\frac{\cos n\psi}{A_n}\right\}. \tag{2}$$

This formula is not restricted to frequencies less than $f_{\max}$. In fact, $f_{\max}$ is not mentioned in [3]. In [3, Sec. 2], Wu states that the series in (2) converges for every ψ lying in the open interval $(0, 2\pi)$, but not at $\psi = 0$ and $\psi = 2\pi$. At the end of the section “Analytical formulation” of [5], a physical interpretation of the lack of convergence at $\psi = 0$ and $\psi = 2\pi$ is provided. However, an appropriate mathematical explanation of the convergence of the series over $(0, 2\pi)$, and of its lack of convergence at $\psi = 0$ and $\psi = 2\pi$, is neither present in [3] nor in [5].

The distribution of the current actually used in [1]–[2], [5], [6, Ch. 4] and [7, Sec. 11.4] is

$$i_{\mathrm{E}}(\psi) = \frac{V_0}{j\pi\eta}\left\{\frac{1}{A_0} + 2\sum_{n=1}^{N}\frac{\cos n\psi}{A_n}\right\}, \tag{3}$$

where the positive integer N is chosen to be “sufficiently large but not too large”, the value $N = 20$ being considered sufficient for any frequency less than $f_{\max}$. Furthermore, observing that the conventional admittance Y_δ defined by

$$Y_\delta = \frac{1}{j\pi\eta}\left\{\frac{1}{A_0} + 2\sum_{n=1}^{N}\frac{1}{A_n}\right\} \tag{4}$$

as a consequence of (3) does not take into account the actual configuration close to the terminals of a real loop antenna, these documents assert that, to obtain a good measure of the impedance of the loop antenna, denoted by Z_{ant}, the conventional admittance Y_δ should be corrected to take into account this actual configuration by introducing a suitable terminal-zone network in the model.

This article investigates (1)–(4), terminal-zone networks and Z_{ant}. It is organized as follows. The Wu-King factors are analyzed from Section II to Section V. The current distribution is discussed in Section VI and Section VII. The conventional admittance Y_δ, terminal-zone networks and Z_{ant} are discussed in Section VIII.

Copyright © 2026 by Excem

II. THE WU-KING FACTORS

A. ASSUMPTIONS AND DEFINITIONS

The single-turn circular wire loop antenna is similar to the one shown in Fig. 1. It is made of a PEC having a circular cross-section of diameter d_{w}. The center line of the PEC is an arc of a circle of radius a, in the plane $z = 0$. The loop antenna is used for emission or reception of time-harmonic signals at a radian frequency $\omega > 0$, corresponding to a frequency f and to a wave number k in said medium. A time factor $e^{j\omega t}$ is assumed and suppressed throughout the paper.

We assume that: the wire is sufficiently thin for $d_{\mathrm{w}} \ll 2a$ to hold; and the wire radius is electrically small, that is, $kd_{\mathrm{w}} \ll 1$. The antenna's positive and negative terminals correspond to the angles $\psi_{\mathrm{min}} > 0$ and $\psi_{\mathrm{max}} = 2\pi - \psi_{\mathrm{min}}$ shown in Fig. 1. The physical space between these terminals is the gap. Since we postulate a narrow gap, $\psi_{\mathrm{min}} \ll \pi/18$.

The Wu-King factors are frequency-dependent dimensionless complex numbers, given by (5)–(7) shown at the bottom of this page [1, Sec. III], in which I_0 and K_0 are modified Bessel functions. For r and s lying in $\mathbb{N}$, where $s \geqslant 1$, the coefficients $\chi_{s\,r}$ showing up in (5)–(7) are dimensionless numbers that are independent of the frequency, a and d_{w}, and are such that: if s is even, then $\chi_{s\,r}$ is real and given by

$$\chi_{s\,r} = \frac{(-1)^{\frac{s}{2}+r}}{s\,\Gamma\!\left(\dfrac{s}{2}+r+0.5\right)\Gamma\!\left(\dfrac{s}{2}-r+0.5\right)}\,; \tag{8}$$

whereas, if s is odd, then $\chi_{s\,r}$ is 0 if $s \leqslant 2r-1$, or imaginary and given by

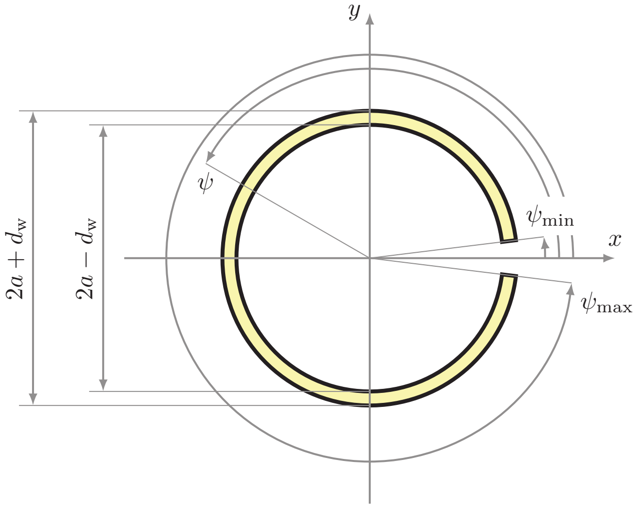

FIGURE 1. The single-turn circular wire loop antenna. In the theoretical case of a delta-gap source, $\psi_{\mathrm{min}} \to 0$ and $\psi_{\mathrm{max}} \to 2\pi$.

$$\chi_{s\,r} = -j\,\frac{(-1)^{\frac{s-1}{2}-r}}{s\left(\dfrac{s-1}{2}+r\right)!\left(\dfrac{s-1}{2}-r\right)!} \tag{9}$$

if $s \geqslant 2r+1$. In (5)–(7), for a positive integer p, the coefficient Ξ_p is a dimensionless real number given by

$$\Xi_p = \ln(4p) + \gamma - 2\sum_{m=0}^{p-1}\frac{1}{2m+1}\,, \tag{10}$$

where γ is Euler's constant.

$$A_0 = ka\,\frac{\mathrm{K}_0\!\left(\dfrac{d_{\mathrm{w}}}{2a}\right)\mathrm{I}_0\!\left(\dfrac{d_{\mathrm{w}}}{2a}\right)+\Xi_1}{\pi} + \sum_{q=2}^{\infty}\chi_{(q-1)\,1}(ka)^q\,, \tag{5}$$

$$A_1 = -\frac{1}{ka}\,\frac{\mathrm{K}_0\!\left(\dfrac{d_{\mathrm{w}}}{2a}\right)\mathrm{I}_0\!\left(\dfrac{d_{\mathrm{w}}}{2a}\right)+\Xi_1}{\pi} + ka\left[\frac{\mathrm{K}_0\!\left(\dfrac{d_{\mathrm{w}}}{a}\right)\mathrm{I}_0\!\left(\dfrac{d_{\mathrm{w}}}{a}\right)+\ln\dfrac{16a}{d_{\mathrm{w}}}+\Xi_2}{2\pi} - \chi_{2\,1}\right] + \sum_{q=2}^{\infty}\left(\frac{\chi_{(q-1)\,0}+\chi_{(q-1)\,2}}{2} - \chi_{(q+1)\,1}\right)(ka)^q\,, \tag{6}$$

and, for $n \in \mathbb{N}$ such that $n \geqslant 2$,

$$\begin{aligned} A_n = {} & -\frac{n^2}{ka}\,\frac{\mathrm{K}_0\!\left(\dfrac{nd_{\mathrm{w}}}{2a}\right)\mathrm{I}_0\!\left(\dfrac{nd_{\mathrm{w}}}{2a}\right)+\Xi_n}{\pi} + \\ & ka\left[\frac{\mathrm{K}_0\!\left(\dfrac{(n+1)d_{\mathrm{w}}}{2a}\right)\mathrm{I}_0\!\left(\dfrac{(n+1)d_{\mathrm{w}}}{2a}\right)+\mathrm{K}_0\!\left(\dfrac{(n-1)d_{\mathrm{w}}}{2a}\right)\mathrm{I}_0\!\left(\dfrac{(n-1)d_{\mathrm{w}}}{2a}\right)+\Xi_{n+1}+\Xi_{n-1}}{2\pi} - n^2\chi_{2\,n}\right] + \\ & \sum_{q=2}^{\infty}\left(\frac{\chi_{(q-1)\,(n-1)}+\chi_{(q-1)\,(n+1)}}{2} - n^2\chi_{(q+1)\,n}\right)(ka)^q\,. \end{aligned} \tag{7}$$

Copyright ©2026 by Excem

TABLE 1. Some approximate values of $\chi_{s\,r}$ defined in Section II.A. Note that Table 1 of [1] contains some typos.

r s	0	1	2	3	4	5	6
1	$-1.000000\,j$	0	0	0	0	0	0
2	-6.366198×10^{-1}	2.122066×10^{-1}	4.244132×10^{-2}	1.818914×10^{-2}	1.010508×10^{-2}	6.430503×10^{-3}	4.451887×10^{-3}
3	$3.333333 \times 10^{-1}\,j$	$-1.666667 \times 10^{-1}\,j$	0	0	0	0	0
4	1.414711×10^{-1}	-8.488264×10^{-2}	1.212609×10^{-2}	1.347343×10^{-3}	3.674573×10^{-4}	1.413297×10^{-4}	6.595387×10^{-5}
5	$-5.000000 \times 10^{-2}\,j$	$3.333333 \times 10^{-2}\,j$	$-8.333333 \times 10^{-3}\,j$	0	0	0	0
6	-1.509025×10^{-2}	1.077875×10^{-2}	-3.592916×10^{-3}	3.266287×10^{-4}	2.512529×10^{-5}	5.025057×10^{-6}	1.477958×10^{-6}
7	$3.968254 \times 10^{-3}\,j$	$-2.976190 \times 10^{-3}\,j$	$1.190476 \times 10^{-3}\,j$	$-1.984127 \times 10^{-4}\,j$	0	0	0
8	9.238926×10^{-4}	-7.185832×10^{-4}	3.266287×10^{-4}	-7.537586×10^{-5}	5.025057×10^{-6}	2.955916×10^{-7}	4.667236×10^{-8}
9	$-1.929012 \times 10^{-4}\,j$	$1.543210 \times 10^{-4}\,j$	$-7.716049 \times 10^{-5}\,j$	$2.204586 \times 10^{-5}\,j$	$-2.755732 \times 10^{-6}\,j$	0	0
10	-3.649946×10^{-5}	2.986320×10^{-5}	-1.608018×10^{-5}	5.360061×10^{-6}	-9.458931×10^{-7}	4.978385×10^{-8}	2.370659×10^{-9}
15	$2.624507 \times 10^{-9}\,j$	$-2.296443 \times 10^{-9}\,j$	$1.530962 \times 10^{-9}\,j$	$-7.654811 \times 10^{-10}\,j$	$2.783568 \times 10^{-10}\,j$	$-6.958919 \times 10^{-11}\,j$	$1.070603 \times 10^{-11}\,j$
20	3.893111×10^{-14}	$-3.522339 \times 10^{-14}$	2.603468×10^{-14}	$-1.562081 \times 10^{-14}$	7.521129×10^{-15}	$-2.852842 \times 10^{-15}$	8.282444×10^{-16}
25	$-1.743356 \times 10^{-19}\,j$	$1.609252 \times 10^{-19}\,j$	$-1.264412 \times 10^{-19}\,j$	$8.429413 \times 10^{-20}\,j$	$-4.741545 \times 10^{-20}\,j$	$2.231315 \times 10^{-20}\,j$	$-8.677337 \times 10^{-21}\,j$
30	$-2.973087 \times 10^{-25}$	2.781275×10^{-25}	$-2.275589 \times 10^{-25}$	1.625421×10^{-25}	$-1.010397 \times 10^{-25}$	5.440597×10^{-26}	$-2.521252 \times 10^{-26}$

TABLE 2. Some approximate values of $\alpha_{q\,n}$ defined in Section II.B.

n q	0	1	2	3	4	5	6
2	0	$-3.333333 \times 10^{-1}\,j$	0	0	0	0	0
3	2.122066×10^{-1}	-2.122066×10^{-1}	6.669350×10^{-2}	1.414711×10^{-2}	6.430503×10^{-3}	3.745238×10^{-3}	2.473270×10^{-3}
4	$-1.666667 \times 10^{-1}\,j$	$1.333333 \times 10^{-1}\,j$	$-5.000000 \times 10^{-2}\,j$	0	0	0	0
5	-8.488264×10^{-2}	6.601983×10^{-2}	-2.739598×10^{-2}	3.307116×10^{-3}	3.423320×10^{-4}	9.107916×10^{-5}	3.491676×10^{-5}
6	$3.333333 \times 10^{-2}\,j$	$-2.619048 \times 10^{-2}\,j$	$1.190476 \times 10^{-2}\,j$	$-2.380952 \times 10^{-3}\,j$	0	0	0
7	1.077875×10^{-2}	-8.622998×10^{-3}	4.246173×10^{-3}	-1.105513×10^{-3}	8.542597×10^{-5}	5.911832×10^{-6}	1.104579×10^{-6}
8	$-2.976190 \times 10^{-3}\,j$	$2.425044 \times 10^{-3}\,j$	$-1.278660 \times 10^{-3}\,j$	$3.968254 \times 10^{-4}\,j$	$-5.511464 \times 10^{-5}\,j$	0	0
9	-7.185832×10^{-4}	5.953975×10^{-4}	-3.326588×10^{-4}	1.175863×10^{-4}	-2.240584×10^{-5}	1.291269×10^{-6}	6.800829×10^{-8}
10	$1.543210 \times 10^{-4}\,j$	$-1.297699 \times 10^{-4}\,j$	$7.615841 \times 10^{-5}\,j$	$-2.981201 \times 10^{-5}\,j$	$7.014590 \times 10^{-6}\,j$	$-7.515633 \times 10^{-7}\,j$	0
15	1.767903×10^{-8}	-1.563914×10^{-8}	1.078257×10^{-8}	-5.719686×10^{-9}	2.277404×10^{-9}	$-6.518086 \times 10^{-10}$	1.239047×10^{-10}
20	$-3.597186 \times 10^{-13}\,j$	$3.273629 \times 10^{-13}\,j$	$-2.463008 \times 10^{-13}\,j$	$1.523683 \times 10^{-13}\,j$	$-7.674315 \times 10^{-14}\,j$	$3.098475 \times 10^{-14}\,j$	$-9.792538 \times 10^{-15}\,j$
25	$-2.047053 \times 10^{-18}$	1.896434×10^{-18}	$-1.506481 \times 10^{-18}$	1.023244×10^{-18}	$-5.913369 \times 10^{-19}$	2.885923×10^{-19}	$-1.176765 \times 10^{-19}$
30	$4.234694 \times 10^{-24}\,j$	$-3.971246 \times 10^{-24}\,j$	$3.273451 \times 10^{-24}\,j$	$-2.367786 \times 10^{-24}\,j$	$1.498651 \times 10^{-24}\,j$	$-8.265260 \times 10^{-25}\,j$	$3.949037 \times 10^{-25}\,j$

Some approximate values of $\chi_{s\,r}$ are shown in Table 1. Alternative formulas for the computation of the coefficients $\chi_{s\,r}$ are obtained in Appendix A.

B. THE POWER SERIES

For any $n \in \mathbb{N}$, a power series $\mathscr{S}_n(ka)$ in the variable ka is present in (5) if $n = 0$, or (6) if $n = 1$, or (7) if $n \geqslant 2$. It has no constant term and starts with a term proportional to ka.

An approximation of A_n may be defined by $d_{\mathrm{A}}(n) \in \mathbb{N}$ such that $\mathscr{S}_n(ka)$ is computed as the sum of its terms of degree $d \leqslant d_{\mathrm{A}}(n)$ in ka. Alternatively, an approximation of A_n may be defined by $d_{\mathrm{B}}(n) \in \mathbb{N}$ such that $\mathscr{S}_n(ka)$ is computed as the sum obtained by replacing each $\chi_{s\,r}$ such that $s > d_{\mathrm{B}}(n)$ with 0, in (5)–(7).

For any integer $q \geqslant 2$, we now define

$$\alpha_{q\,0} = \chi_{(q-1)\,1} \tag{11}$$

and, for any positive integer n,

$$\alpha_{q\,n} = \frac{\chi_{(q-1)\,(n-1)} + \chi_{(q-1)\,(n+1)}}{2} - n^2 \chi_{(q+1)\,n}\,. \tag{12}$$

Some approximate values of $\alpha_{q\,n}$ are shown in Table 2. It follows from (8)–(9) and (11)–(12) that:

- for any $n \in \mathbb{N}$ and any integer $q \geqslant 2$, the coefficient $\alpha_{q\,n}$ is a dimensionless number that is independent of the frequency, a and d_{w} ;
- for any $n \in \mathbb{N}$ and any integer $q \geqslant 2$, the coefficient $\alpha_{q\,n}$ is real if q is odd, or imaginary if q is even;
- we have $\alpha_{2\,0} = 0$ because $\chi_{1\,1} = 0$;
- we have $\alpha_{2\,1} = -j/3$, since $\chi_{1\,0} = -j$, $\chi_{1\,2} = 0$ and $\chi_{3\,1} = -j/6$; and
- for any positive integers p and n such that $n \geqslant p + 1$, we have $\alpha_{(2p)\,n} = 0$.

For any $n \in \mathbb{N}$, the power series

$$\mathrm{S}_n(ka) = \sum_{q=2}^{\infty} \alpha_{q\,n} (ka)^q \tag{13}$$

in the real variable ka is present in the formula that, among (5)–(7), defines A_n. Except for its first term, the series $\mathscr{S}_n(ka)$ is identical to $\mathrm{S}_n(ka)$. Unlike $\mathscr{S}_n(ka)$, though, $\mathrm{S}_n(ka)$ does not depend on $d_{\mathrm{w}}/(2a)$.

Copyright ©2026 by Excem

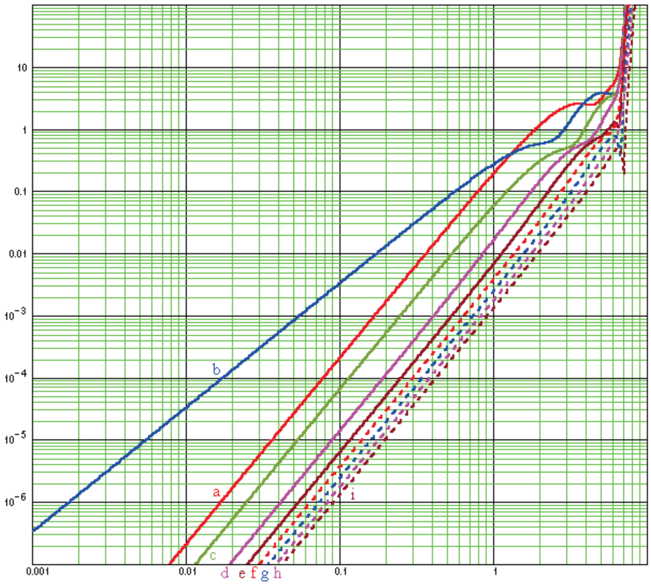

FIGURE 2. $\mathcal{S}_n(ka)$ as a function of ka, computed for $d_\mathrm{A}(n) = 30$. $\mathcal{S}_0(ka)$ is curve "a", $\mathcal{S}_1(ka)$ is curve "b", $\mathcal{S}_2(ka)$ is curve "c", $\mathcal{S}_3(ka)$ is curve "d", $\mathcal{S}_4(ka)$ is curve "e", $\mathcal{S}_5(ka)$ is curve "f", $\mathcal{S}_6(ka)$ is curve "g", $\mathcal{S}_7(ka)$ is curve "h", and $\mathcal{S}_8(ka)$ is curve "i".

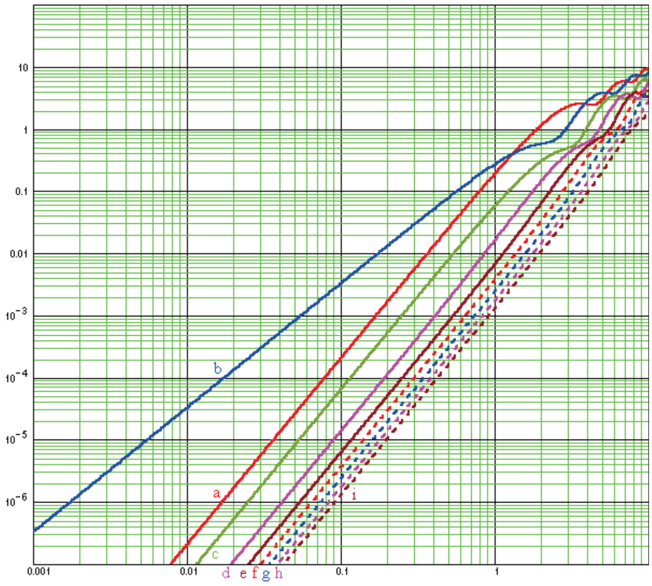

FIGURE 3. $\mathcal{S}_n(ka)$ as a function of ka, computed for $d_\mathrm{A}(n) = 60$. $\mathcal{S}_0(ka)$ is curve "a", $\mathcal{S}_1(ka)$ is curve "b", $\mathcal{S}_2(ka)$ is curve "c", $\mathcal{S}_3(ka)$ is curve "d", $\mathcal{S}_4(ka)$ is curve "e", $\mathcal{S}_5(ka)$ is curve "f", $\mathcal{S}_6(ka)$ is curve "g", $\mathcal{S}_7(ka)$ is curve "h", and $\mathcal{S}_8(ka)$ is curve "i".

It is shown in Appendix B that the radius of convergence of $\mathrm{S}_n(ka)$ is infinity. Thus, the radius of convergence of $\mathscr{S}_n(ka)$ is infinity.

For any $n \in \mathbb{N}$ and a specified $d_\mathrm{A}(n) \geqslant 2$, an approximation of $\mathrm{S}_n(ka)$ is the polynomial $\mathcal{S}_n(ka)$ given by

$$\mathcal{S}_n(ka) = \sum_{q=2}^{d_\mathrm{A}(n)} \alpha_{q\,n} (ka)^q \,. \tag{14}$$

To discuss the accuracy of this approximation, we consider the absolute error

$$\mathcal{E}_n(ka) = \mathcal{S}_n(ka) - \mathrm{S}_n(ka) = -\sum_{q=d_\mathrm{A}(n)+1}^{\infty} \alpha_{q\,n} (ka)^q \,, \tag{15}$$

and, for any integer $q \geqslant 1$, we define

$$\mathfrak{E}_n(ka, q) = \sqrt{\begin{matrix} \left[|\alpha_{(q+1)\,n}| \, (ka)^{q+1} \right]^2 \\ + \left[|\alpha_{(q+2)\,n}| \, (ka)^{q+2} \right]^2 \end{matrix}} \,. \tag{16}$$

Let $\mathrm{ceil}(y)$ be the smallest integer greater than or equal to a real number y. Appendix C shows that, if

$$d_\mathrm{A}(0) \geqslant 2 \,\mathrm{ceil}\left(\sqrt{(ka)^2 + 1} \right) , \tag{17}$$

in the case $n = 0$, or if

$$d_\mathrm{A}(n) \geqslant \max\left(2 \,\mathrm{ceil}\left(\sqrt{4(ka)^2 + (n-1)^2} \right), 2n+2 \right), \tag{18}$$

in the case $n \geqslant 1$, then, we have

$$|\mathcal{E}_n(ka)| \leqslant \mathfrak{E}_n\left(ka, d_\mathrm{A}(n)\right) \,. \tag{19}$$

$\mathcal{S}_n(ka)$ is shown as a function of ka for $n \in \{0, \ldots, 8\}$, computed for $d_\mathrm{A}(n) = 30$ in Fig. 2 and for $d_\mathrm{A}(n) = 60$ in Fig. 3. We observe that the curves in Fig. 2 differ markedly from the corresponding ones in Fig. 3 for $ka \geqslant 6$.

For $ka = 2.5$ and $n \leqslant 8$, the value $d_\mathrm{A}(n) = 30$ satisfies the conditions (17)–(18). Thus, it follows from (19) that $\mathfrak{E}_n(2.5, 30)$ is an upper bound on $|\mathcal{E}_n(2.5)|$ for $n \leqslant 8$ and $d_\mathrm{A}(n) = 30$. Using this, we find that $|\mathcal{E}_n(ka)| \leqslant 6.2 \times 10^{-13}$ for $ka \leqslant 2.5$, $n \leqslant 8$ and $d_\mathrm{A}(n) = 30$. Thus, Fig. 2 is very accurate at least up to $ka = 2.5$.

For $ka = 5.5$ and $n \leqslant 8$, the value $d_\mathrm{A}(n) = 30$ satisfies the conditions (17)–(18). Thus, $\mathfrak{E}_n(5.5, 30)$ is an upper bound on $|\mathcal{E}_n(5.5)|$ for $n \leqslant 8$ and $d_\mathrm{A}(n) = 30$. Using this, we find that $|\mathcal{E}_n(ka)| \leqslant 0.027$ for $ka \leqslant 5.5$, $n \leqslant 8$ and $d_\mathrm{A}(n) = 30$. It follows that Fig. 2 is reasonably accurate at least up to $ka = 5.5$.

For $ka = 5.5$ and $n \leqslant 20$, the value $d_\mathrm{A}(n) = 60$ satisfies the conditions (17)–(18). Thus, $\mathfrak{E}_n(5.5, 60)$ is an upper bound on $|\mathcal{E}_n(5.5)|$ for $n \leqslant 20$ and $d_\mathrm{A}(n) = 60$. Using this, we find that $|\mathcal{E}_n(ka)| \leqslant 1.1 \times 10^{-20}$ for $ka \leqslant 5.5$, $n \leqslant 20$ and $d_\mathrm{A}(n) = 60$.

Using 100 values of ka per decade, an evaluation of the relative error of $\mathcal{S}_n(ka)$ computed for $d_\mathrm{A}(n) = 30$, regarded as an approximation of $\mathcal{S}_n(ka)$ computed for $d_\mathrm{A}(n) = 60$, shows that, for $n \leqslant 20$, the absolute value of this relative error is less than 10^{-8} up to $ka = 2.5$, and less than 0.9% up to $ka = 5.5$.

For $ka = 10$ and $n \leqslant 20$, the value $d_\mathrm{A}(n) = 60$ satisfies the conditions (17)–(18). Thus, $\mathfrak{E}_n(10, 60)$ is an upper bound on $|\mathcal{E}_n(10)|$ for $n \leqslant 20$ and $d_\mathrm{A}(n) = 60$. Using this, we find that $|\mathcal{E}_n(ka)| \leqslant 7.3 \times 10^{-5}$ for $ka \leqslant 10$, $n \leqslant 20$ and $d_\mathrm{A}(n) = 60$. It follows that Fig. 3 is very accurate.

We have shown that $\mathrm{S}_n(ka)$ can be computed with any desired accuracy, at frequencies far beyond $f_{\max}$ given by (1), which corresponds to $ka = 2.5$.

Copyright ©2026 by Excem

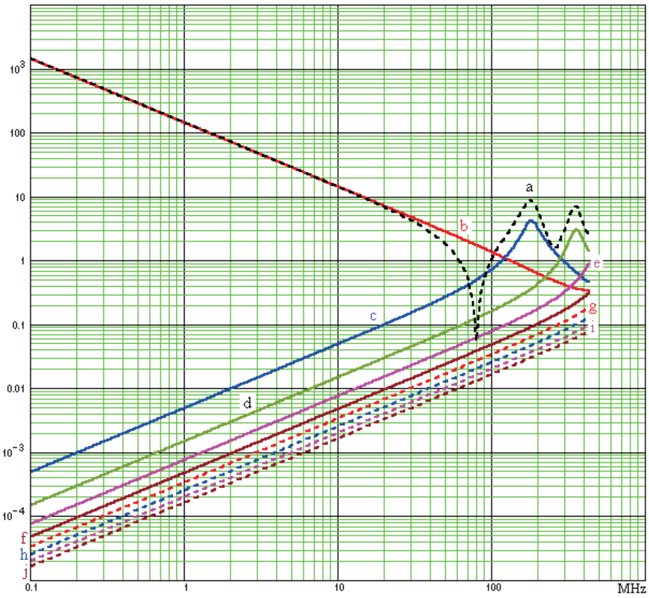

FIGURE 4. Information on the Wu-King factors. The absolute value of the contents of the curly brackets in (4) for $N = 20$ is curve "a". $|1/A_0|$ is curve "b", $|1/A_1|$ is curve "c", $|1/A_2|$ is curve "d", $|1/A_3|$ is curve "e", $|1/A_4|$ is curve "f", $|1/A_5|$ is curve "g", $|1/A_6|$ is curve "h", $|1/A_7|$ is curve "i", and $|1/A_8|$ is curve "j".

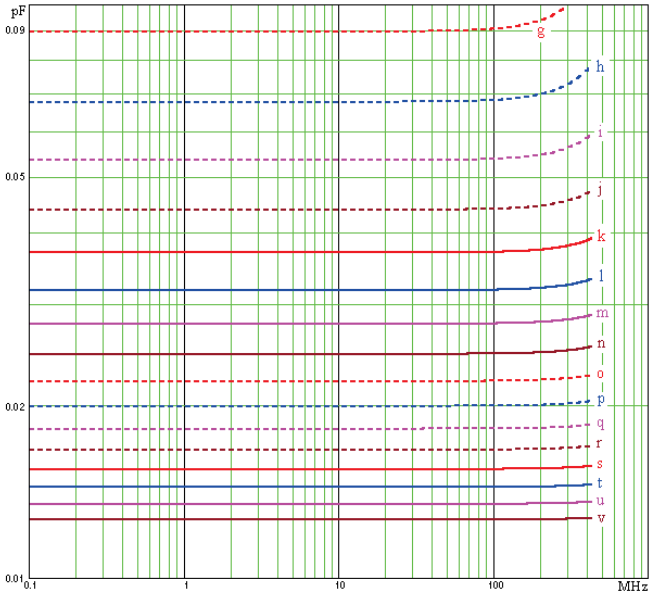

FIGURE 5. $|\mathcal{C}_{\mathrm{R}}/A_5|$ is curve "g", $|\mathcal{C}_{\mathrm{R}}/A_6|$ is curve "h", $|\mathcal{C}_{\mathrm{R}}/A_7|$ is curve "i", $|\mathcal{C}_{\mathrm{R}}/A_8|$ is curve "j", $|\mathcal{C}_{\mathrm{R}}/A_9|$ is curve "k", $|\mathcal{C}_{\mathrm{R}}/A_{10}|$ is curve "l", $|\mathcal{C}_{\mathrm{R}}/A_{11}|$ is curve "m", $|\mathcal{C}_{\mathrm{R}}/A_{12}|$ is curve "n", $|\mathcal{C}_{\mathrm{R}}/A_{13}|$ is curve "o", $|\mathcal{C}_{\mathrm{R}}/A_{14}|$ is curve "p", $|\mathcal{C}_{\mathrm{R}}/A_{15}|$ is curve "q", $|\mathcal{C}_{\mathrm{R}}/A_{16}|$ is curve "r", $|\mathcal{C}_{\mathrm{R}}/A_{17}|$ is curve "s", $|\mathcal{C}_{\mathrm{R}}/A_{18}|$ is curve "t", $|\mathcal{C}_{\mathrm{R}}/A_{19}|$ is curve "u", and $|\mathcal{C}_{\mathrm{R}}/A_{20}|$ is curve "v".

C. COMPACT FORMULATIONS

Some additional definitions are needed to obtain compact formulations of the Wu-King factors. For any positive integer n, we define the capacitance

$$\mathfrak{C}_n = \frac{2\epsilon a}{n^2} \, \frac{1}{\mathrm{K}_0\left(\dfrac{n d_{\mathrm{w}}}{2a}\right) \mathrm{I}_0\left(\dfrac{n d_{\mathrm{w}}}{2a}\right) + \Xi_n} \,. \tag{20}$$

We also define the inductance

$$\mathfrak{L}_0 = \mu a \left[\mathrm{K}_0\left(\frac{d_{\mathrm{w}}}{2a}\right) \mathrm{I}_0\left(\frac{d_{\mathrm{w}}}{2a}\right) + \Xi_1 \right], \tag{21}$$

the inductance

$$\mathfrak{L}_1 = \frac{\mu a}{4} \left[\mathrm{K}_0\left(\frac{d_{\mathrm{w}}}{a}\right) \mathrm{I}_0\left(\frac{d_{\mathrm{w}}}{a}\right) + \ln\frac{16a}{d_{\mathrm{w}}} + \Xi_2 - 2\pi\chi_{2\,1} \right], \tag{22}$$

and, for any $n \in \mathbb{N}$ such that $n \geqslant 2$, the inductance

$$\begin{aligned} \mathfrak{L}_n = \frac{\mu a}{4} \Bigg[& \mathrm{K}_0\left(\frac{(n+1) d_{\mathrm{w}}}{2a}\right) \mathrm{I}_0\left(\frac{(n+1) d_{\mathrm{w}}}{2a}\right) \\ & + \mathrm{K}_0\left(\frac{(n-1) d_{\mathrm{w}}}{2a}\right) \mathrm{I}_0\left(\frac{(n-1) d_{\mathrm{w}}}{2a}\right) \\ & + \Xi_{n+1} + \Xi_{n-1} - 2\pi n^2 \chi_{2\,n} \Bigg] \,. \end{aligned} \tag{23}$$

Let P be the function defined on the set $\mathbb{R}_+^*$ of the positive real numbers by $\forall x \in \mathbb{R}_+^*$, $P(x) = \mathrm{K}_0(x)\,\mathrm{I}_0(x)$. It appears once in (20)–(22), and twice in (23). P is differentiable and, for any $x \in \mathbb{R}_+^*$, $\mathrm{K}_0(x)$ and $\mathrm{I}_0(x)$ are known to be real and positive [8, Sec. 9.6.1], hence $P(x)$ is real and positive.

Thus, it follows from (8) and (10) that the capacitances defined by (20) and the inductances defined by (21)–(23) are real. They are frequency-independent, and such that

$$A_0 = \frac{\omega}{\pi\eta} \mathfrak{L}_0 + \mathrm{S}_0(ka) \,, \tag{24}$$

and, for any positive integer n,

$$A_n = -\frac{2}{\omega\pi\eta\,\mathfrak{C}_n} + \frac{2\omega}{\pi\eta} \mathfrak{L}_n + \mathrm{S}_n(ka) \,. \tag{25}$$

III. EXAMPLE — PART 1

We consider a loop antenna made of a PEC in vacuum, with $a = 280\,\mathrm{mm}$ and $d_{\mathrm{w}} = 14\,\mathrm{mm}$, for which $ka = 2.5$ at $f_{\max} \simeq 426.0\,\mathrm{MHz}$.

The 10 curves shown in Fig. 4 allow us to see that, up to about 30 MHz, $|1/A_0|$ is much greater than $|1/A_n|$ for $n \geqslant 1$, hence the contents of the curly brackets in (4) is very close to $|1/A_0|$. Up to about 100 MHz, $|1/A_0|$ has a slope of about -20 dB/decade, characteristic of the term including $\mathfrak{L}_0$ in (24). Up to about 60 MHz, $|1/A_1|$ to $|1/A_8|$ have a slope of about 20 dB/decade, characteristic of the term including $\mathfrak{C}_n$ in (25). To study this behavior, we introduce the real quantity

$$\mathcal{C}_{\mathrm{R}} = \frac{-2\epsilon}{\pi k} = \frac{-2}{\pi\omega\eta} = \frac{-1}{\pi^2 f \eta} \,, \tag{26}$$

in which ω is the radian frequency and f the frequency.

Let n be a positive integer. $\mathcal{C}_{\mathrm{R}}$ having the dimensions of capacitance, $\mathcal{C}_{\mathrm{R}}/A_n$ is a complex frequency-dependent capacitance, which by (25) tends to $\mathfrak{C}_n$ as $\omega \to 0$.

Fig. 5 shows the absolute values of the 16 capacitances $\mathcal{C}_{\mathrm{R}}/A_5$ to $\mathcal{C}_{\mathrm{R}}/A_{20}$, expressed in pF, up to about 426.6 MHz. We see that, in our example, $\mathcal{C}_{\mathrm{R}}/A_5$ to $\mathcal{C}_{\mathrm{R}}/A_{20}$ are nearly frequency independent up to 200 MHz, and that $\mathcal{C}_{\mathrm{R}}/A_{10}$ to $\mathcal{C}_{\mathrm{R}}/A_{20}$ are nearly frequency independent up to $f_{\max}$.

Copyright ©2026 by Excem

TABLE 3. Computed values of the inductances $\mathfrak{L}_n$, the capacitances $\mathfrak{C}_n$ and $\mathcal{C}_\mathrm{R}/A_n$ at $f_0 = 100$ kHz.

n	$\mathfrak{L}_n$	$\mathfrak{C}_n$	$\mathcal{C}_\mathrm{R}/A_n$ at f_0
0	1326.389 nH		
1	663.355 nH	1.315327 pF	1.315328 pF
2	476.212 nH	0.399072 pF	0.399072 pF
3	396.559 nH	0.203243 pF	0.203243 pF
4	343.911 nH	0.127500 pF	0.127500 pF
5	304.426 nH	0.089547 pF	0.089547 pF
6	272.890 nH	0.067505 pF	0.067505 pF
7	246.724 nH	0.053415 pF	0.053415 pF
8	224.444 nH	0.043780 pF	0.043780 pF
9	205.116 nH	0.036853 pF	0.036853 pF
10	188.109 nH	0.031677 pF	0.031677 pF
11	172.976 nH	0.027689 pF	0.027689 pF
12	159.390 nH	0.024538 pF	0.024538 pF
13	147.101 nH	0.021997 pF	0.021997 pF
14	135.918 nH	0.019911 pF	0.019911 pF
15	125.686 nH	0.018173 pF	0.018173 pF
16	116.281 nH	0.016706 pF	0.016706 pF
17	107.602 nH	0.015453 pF	0.015453 pF
18	99.566 nH	0.014373 pF	0.014373 pF
19	92.101 nH	0.013434 pF	0.013434 pF
20	85.148 nH	0.012610 pF	0.012610 pF

Thus, Fig. 5 and Table 3 shows that, in our example, the approximation

$$A_n \simeq -\frac{2}{\omega\pi\eta\,\mathfrak{C}_n} \tag{27}$$

is valid for $n = 5$ to $n = 20$ up to 200 MHz, and for $n = 10$ to $n = 20$ up to f_max.

The curves shown in Fig. 4 and Fig. 5 were plotted up to about $426.6\,\mathrm{MHz}$ with 100 points per decade of frequency. At these points, and for all $n \in \{0, \ldots 20\}$, the Wu-King factors have been computed a first time using $d_\mathrm{B}(n) = 30$, as in [2, Sec. 3], and a second time using $d_\mathrm{A}(n) = 30$. For each Wu-King factor and at each of these frequency points, the absolute value of the relative difference between these computations is less than 7.1×10^{-13}. It follows from the discussion of Section II.B that the results of both computations are extremely accurate.

IV. FURTHER STUDY OF THE WU-KING FACTORS

A. THE PRODUCT OF BESSEL FUNCTIONS

We wrote in Section II.C that the function P is real and positive. P is strictly decreasing on $\mathbb{R}_+^*$ [9]. For small values of $x \in \mathbb{R}_+^*$, it satisfies

$$P(x) = -\left(\ln\frac{x}{2} + \gamma\right)\left(1 + \frac{x^2}{2}\right) + \frac{x^2}{4} + o\left(x^3\right), \tag{28}$$

which was used in [1, Sec. III.C], whereas, for large values of $x \in \mathbb{R}_+^*$, it satisfies [8, Eq. 9.7.5]

$$P(x) = \frac{1}{2x} + \frac{1}{16x^3} + \frac{27}{128x^5} + O\left(\frac{1}{x^7}\right), \tag{29}$$

in which we have used Landau's little-o and big-O notations [10, Sec. 5.1].

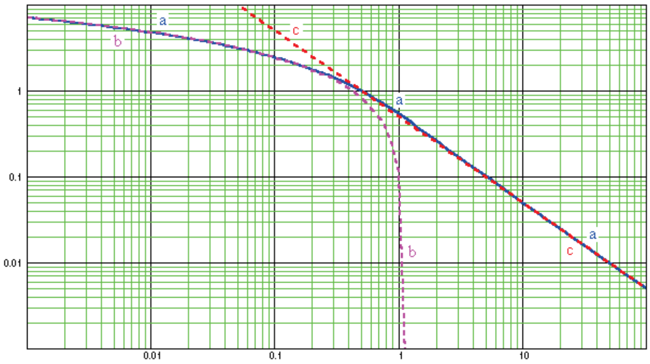

FIGURE 6. $P(x)$ is curve "a". The small-value approximation $P_\mathrm{S}(x)$ is curve "b". The large-value approximation $P_\mathrm{L}(x)$ is curve "c".

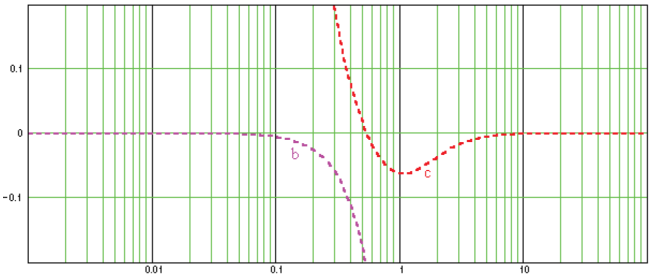

FIGURE 7. The relative error of $P_\mathrm{S}(x)$ as an approximation of $P(x)$ is curve "b". The relative error of $P_\mathrm{L}(x)$ as an approximation of $P(x)$ is curve "c". Curve "b" is also the relative error of $Q_\mathrm{S}(x)$ as an approximation of $Q(x)$, and curve "c" is also the relative error of $Q_\mathrm{L}(x)$ as an approximation of $Q(x)$.

By (28)–(29), $P(x) \to \infty$ as $x \to 0$, and $P(x) \to 0$ as $x \to \infty$. Fig. 6 shows, as a function of x: $P(x)$; the small-value approximation $P_\mathrm{S}(x) = -\ln(x/2) - \gamma$ related to the right-hand side of (28); and the large-value approximation $P_\mathrm{L}(x) = 1/(2x)$ related to the right-hand side of (29).

Fig. 7 shows the relative errors of $P_\mathrm{S}(x)$ and $P_\mathrm{L}(x)$ as approximations of $P(x)$. In Fig. 7, the absolute value of the relative error of $P_\mathrm{L}(x)$ as a large-value approximation of $P(x)$ is less than 10% if $x > 0.375$, and less than 1% if $x > 3.805$.

B. A CLOSER LOOK AT THE COEFFICIENTS Ξ_p

At this point, p being a positive integer, we need to clarify the effect of Ξ_p in (20)–(23). In Appendix D, it is shown that $\Xi_p < 0$, that the sequence $q \mapsto \Xi_q$ defined for $q \geqslant 1$ is strictly increasing, and that, for large values of p,

$$\Xi_p = -\frac{1}{24p^2} + O\left(\frac{1}{p^4}\right) \tag{30}$$

Hence, the sequence $q \mapsto \Xi_q$ tends to 0 as q becomes large.

Let $t = d_\mathrm{w}/(2a)$. The sum $P(pt) + \Xi_p$ appears once in (20), (21) and (22), and it also appears twice in (23). Since, for $q \geqslant 1$, the sequence $q \mapsto P(qt)$ is positive and strictly decreasing, and the sequence $q \mapsto \Xi_q$ is negative and strictly increasing, the behavior of their sum is not obvious.

To address this issue, we can consider the ratio

$$r_q(t) = \frac{|\Xi_q|}{\mathrm{K}_0\,(qt)\,\mathrm{I}_0\,(qt)} \tag{31}$$

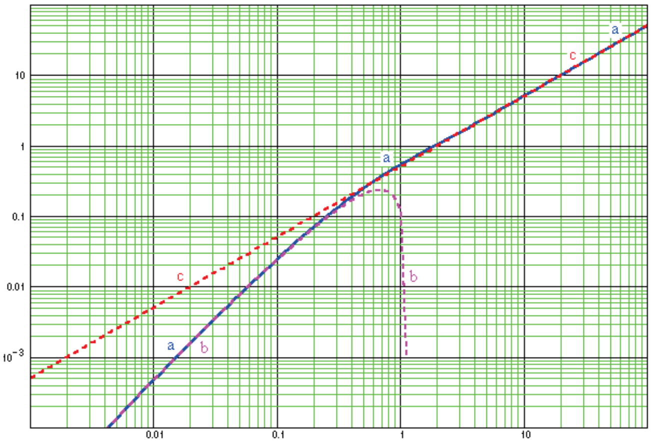

FIGURE 8. $Q(x)$ is curve “a”. The small-value approximation Q_{S} is curve “b”. The large-value approximation Q_{L} is curve “c”.

defined for any positive integer q. In Appendix E, it is shown that the positive function $\tilde{r}$ defined for any $\nu \in \mathbb{R}_+^*$ by

$$\tilde{r}(\nu, t) = \frac{\nu + 2}{12\,\nu^3} \sqrt{(\nu t)^2 + \frac{1}{3}}\,, \tag{32}$$

which approaches 0 as ν becomes large, is strictly decreasing with respect to ν, and such that $r_q(t) < \tilde{r}(q, t)$ for any positive integer q. It follows that $r_q(t) = O(1/q)$.

Since we have assumed $d_{\mathrm{w}} \ll 2a$, we have $t < 1$, hence, for any positive integer q

$$r_q(t) < \tilde{r}(1, t) < \frac{1}{2\sqrt{3}} < \frac{1}{3}\,. \tag{33}$$

Hence,

$$\mathrm{K}_0\,(qt)\,\mathrm{I}_0\,(qt) + \Xi_q > 0\,. \tag{34}$$

If we regard $P(qt)$ as an approximate value of $P(qt) + \Xi_q$, the relative error is $r_q(t)$, because $\Xi_q < 0$. Since $r_q(t)$ is less than $\tilde{r}(q,t)$, (32) can be used to determine a positive integer $q_{\min}(t)$ such that for all integers $q \geqslant q_{\min}(t)$, the relative error $r_q(t)$ is less than or equal to a desired maximum value $r_{\mathrm{MAX}}(t) > 0$. A possible method is: compute $\tilde{r}(1,t)$, to determine if $q_{\min}(t) = 1$ is a possible choice; if not, solve the polynomial equation of degree 6 in the variable ν

$$144\, r_{\mathrm{MAX}}^2(t)\,\nu^6 - t^2\nu^4 - 2t^2\nu^3 - \left(4t^2 + \frac{1}{3}\right)\nu^2 - \frac{4}{3}\nu - \frac{4}{3} = 0 \tag{35}$$

to obtain its positive real root $\nu_{\mathrm{P}}(t)$, which exists and is unique because of the above-mentioned properties of the function $\tilde{r}$; and posit $q_{\min}(t) = \mathrm{ceil}(\nu_{\mathrm{P}}(t))$. Moreover, it is then possible to also compute the least positive integer $q_{\mathrm{MIN}}(t)$ such that, for all integers $q \geqslant q_{\mathrm{MIN}}(t)$, the relative error $r_q(t)$ is less than or equal to $r_{\mathrm{MAX}}(t)$, by computing $r_q(t)$ for integers $q < q_{\min}$ in decreasing order.

For instance, if $t = d_{\mathrm{w}}/(2a) = 1/40$ as in Section III, we get in this manner: if $r_{\mathrm{MAX}}(t) = 10^{-4}$, then $q_{\min}(t) = 29$ and $q_{\mathrm{MIN}}(t) = 21$; if $r_{\mathrm{MAX}}(t) = 10^{-5}$, then $q_{\min}(t) = 212$ and $q_{\mathrm{MIN}}(t) = 208$; and, if $r_{\mathrm{MAX}}(t) = 10^{-6}$, then $q_{\min}(t) = 2086$ and $q_{\mathrm{MIN}}(t) = 2084$.

C. ABOUT THE CAPACITANCES $\mathfrak{C}_n$

By (20) and (34), for any positive integer n, the capacitance $\mathfrak{C}_n$ is positive. P being strictly decreasing on $\mathbb{R}_+^*$, we find that, if the positive integer n is fixed, $\mathfrak{C}_n$ is a strictly increasing function in the variable $t = d_{\mathrm{w}}/(2a) \ll 1$.

We now study the effect of n on $\mathfrak{C}_n$, for a fixed positive value of t. We introduce the function Q defined on $\mathbb{R}_+^*$ by $\forall x \in \mathbb{R}_+^*$, $Q(x) = x^2\,\mathrm{K}_0(x)\,\mathrm{I}_0(x)$, because (20) shows that the effect of n on $\mathfrak{C}_n$ only depends on $D_n(t)$ given by

$$D_n(t) = n^2[P(nt) + \Xi_n] = \frac{1}{t^2}\,Q(nt) + n^2\,\Xi_n\,. \tag{36}$$

The function Q is real and positive, and it is shown in Appendix F that Q is strictly increasing on $\mathbb{R}_+^*$. By (28)–(29), $Q(x) \to 0$ as $x \to 0$, and $Q(x) \to \infty$ as $x \to \infty$. Fig. 8 shows, as a function of x: $Q(x)$; the small-value approximation $Q_{\mathrm{S}}(x) = -x^2(\ln(x/2) + \gamma)$ related to the right-hand side of (28); and the large-value approximation $Q_{\mathrm{L}}(x) = x/2$ related to the right-hand side of (29). Note that Fig. 7 is applicable to $Q_{\mathrm{S}}(x)$ and $Q_{\mathrm{L}}(x)$, because the relative errors of $Q_{\mathrm{S}}(x)$ and $Q_{\mathrm{L}}(x)$ as approximations of $Q(x)$ are equal to the relative errors of $P_{\mathrm{S}}(x)$ and $P_{\mathrm{L}}(x)$ as approximations of $P(x)$, respectively.

By (34), $D_n(t)$ is positive. If we regard $Q(nt)/t^2$ as an approximate value of $D_n(t)$, the relative error is $r_n(t)$ investigated in Section IV.B. Using $r_n(t) = O(1/n)$ and $Q(nt)/t^2 \to \infty$ as $n \to \infty$, we find that $D_n(t) \to \infty$ as $n \to \infty$. It follows from (20) that the sequence $n \mapsto \mathfrak{C}_n$ is real, positive, and such that $\mathfrak{C}_n \to 0$ as $n \to \infty$.

The facts that the sequence $n \mapsto Q(nt)/t^2$ is increasing and $r_n(t) = O(1/n)$ neither allow us to conclude that the sequence $n \mapsto D_n(t)$ is increasing, nor to use (20) to state that the sequence $n \mapsto \mathfrak{C}_n$ is decreasing. However, it is shown in Appendix G that, if $t \leqslant 1/3$, then the sequence $n \mapsto D_n(t)$ is strictly increasing.

Thus, it follows from (20) that, if $t \leqslant 1/3$, then the sequence $n \mapsto \mathfrak{C}_n$ is real, positive, strictly decreasing and such that $\mathfrak{C}_n \to 0$ as $n \to \infty$. This is in line with what is observed in Table 3 and the left-hand side of Fig. 5.

D. ABOUT THE INDUCTANCES $\mathfrak{L}_n$

For $n \geqslant 1$ and any fixed positive value of $t = d_{\mathrm{w}}/(2a) \ll 1$, using (89) and (93) of Appendix A, we obtain

$$\chi_{2\,n} = \frac{2}{\pi(2n-1)(2n+1)}\,, \tag{37}$$

which may be used in (22)–(23).

By (21) and (34), the inductance $\mathfrak{L}_0$ is positive. However, for $n \geqslant 1$, the inductance $\mathfrak{L}_n$ need not be positive, even though this fact does not show up in Table 3. More precisely, it follows from (23), (37), $r_n(t) = O(1/n)$, and from $P(nt) \to 0$ as $n \to \infty$ that, for any fixed positive value of $t = d_{\mathrm{w}}/(2a)$, we have

$$\lim_{n\to\infty} \mathfrak{L}_n = -\frac{\mu a}{4}\,. \tag{38}$$

The function P being strictly decreasing on $\mathbb{R}_+^*$, we find that, if the nonnegative integer n is fixed, $\mathfrak{L}_n$ is a strictly decreasing function in the variable $t = d_{\mathrm{w}}/(2a)$.

Copyright ©2026 by Excem

V. ASYMPTOTIC BEHAVIOR

For a fixed value of the variable $t = d_\mathrm{w}/(2a) \ll 1$, we now want to obtain convenient asymptotic equivalents for the capacitance $\mathfrak{C}_n$, the inductance $\mathfrak{L}_n$ and the Wu-King factor A_n, as $n \to \infty$. Using (29) and (30), we get

$$\mathrm{K}_0\left(\frac{nd_\mathrm{w}}{2a}\right)\mathrm{I}_0\left(\frac{nd_\mathrm{w}}{2a}\right) + \Xi_n = \frac{a}{nd_\mathrm{w}} - \frac{1}{24n^2} + O\left(\frac{1}{n^3}\right). \tag{39}$$

It follows from $d_\mathrm{w} \ll 2a$ that, for any positive n,

$$\frac{1}{24n^2} \ll \frac{1}{12n}\ \frac{a}{nd_\mathrm{w}}\,, \tag{40}$$

hence, in the right-hand side of (39), the absolute value of the second term is much less than the first term. Thus, using (39) and (40) in (20), we obtain

$$\mathfrak{C}_n = \frac{2\epsilon d_\mathrm{w}}{n}\left(1 + \frac{d_\mathrm{w}}{24na}\right) + O\left(\frac{1}{n^3}\right) \tag{41}$$

as n becomes large.

Using (37) and (39) in (23), we find that

$$\mathfrak{L}_n = \frac{\mu a}{4}\left(-1 + \frac{2a}{nd_\mathrm{w}} - \frac{1}{3n^2}\right) + O\left(\frac{1}{n^3}\right) \tag{42}$$

as n becomes large.

The series $\mathrm{S}_n(ka)$ are studied in Appendix H. According to (255), if the condition $f \leqslant f_\mathrm{max}$ is satisfied, then

$$\mathrm{S}_n(ka) = O\left(\frac{1}{n}\right) \tag{43}$$

as n becomes large. Using (37), (39) and (43) in (7), we get

$$A_n = -\frac{n}{\pi k d_\mathrm{w}} + \frac{1}{24\pi ka} - \frac{ka}{2\pi} + O\left(\frac{1}{n}\right) \tag{44}$$

as n becomes large, if $f \leqslant f_\mathrm{max}$. We observe that, in the right-hand side of (44):

- the first term stems from $a/(nd_\mathrm{w})$ as an approximation of $P(nd_\mathrm{w}/(2a))$ in (39), the accuracy of which was discussed in Section IV.A and depicted in Fig. 7;
- the absolute value of the relative error of this approximation is less than 10% if $n > 0.375 \times 2a/d_\mathrm{w}$, and less than 1% if $n > 3.805 \times 2a/d_\mathrm{w}$;
- accordingly, for the example of Section III, the absolute value of the relative error of this first term is less than 10% if $n > 15$, and less than 1% if $n > 152$;
- the second term stems from an approximation of Ξ_n studied in Appendix D, which becomes accurate for moderate values of n, and which by $d_\mathrm{w} \ll 2a$ is much less than $1/(12\pi k d_\mathrm{w})$ for any positive n ;
- the third term stems from an approximation equivalent to using $-\mu a/4$ as an approximation of $\mathfrak{L}_n$, hence according to (42) the absolute value of the relative error of this approximation is less than about 11% if $n > 10 \times 2a/d_\mathrm{w}$; and
- accordingly, for the example of Section III, the absolute value of the relative error of this third term is less than about 11% if $n > 400$.

VI. CURRENT DISTRIBUTION

We now wish to estimate the current distribution $i_\mathrm{W}(\psi)$ defined by (2). We assume $f \leqslant f_\mathrm{max}$, hence $ka \leqslant 2.5$. Since $d_\mathrm{w}/(2a) \ll 1$, we obtain

$$\frac{kd_\mathrm{w}}{n}\left|\frac{ka}{2} - \frac{1}{24ka}\right| \ll 1 \tag{45}$$

for $n \geqslant 7$. Thus, if $f \leqslant f_\mathrm{max}$, it follows from (44) that

$$\frac{1}{A_n} = -\frac{\pi k d_\mathrm{w}}{n} + \frac{\pi (kd_\mathrm{w})^2}{n^2}\left(\frac{ka}{2} - \frac{1}{24\,ka}\right) + O\left(\frac{1}{n^3}\right) \tag{46}$$

as n becomes large. What was said at the end of Section V about the terms of the right-hand side of (44) applies to the corresponding terms of the right-hand side of (46). In particular, if we use x to denote $nd_\mathrm{w}/(2a)$, the first term of the right-hand side of (46) stems from $P_\mathrm{L}(x) = a/(nd_\mathrm{w})$ as a large-value approximation of $P(x) = P(nd_\mathrm{w}/(2a))$ in (39), the accuracy of which was discussed in Section IV.A and depicted in Fig. 7.

It follows from (46) that, for a given value of $ka \leqslant 2.5$, there exist an integer $N_\mathrm{A} \geqslant 6$ and a positive real number M_A such that, for any integer $n > N_\mathrm{A}$, the sequence

$$r_n^\mathrm{A} = \frac{1}{A_n} + \frac{\pi k d_\mathrm{w}}{n} - \frac{\pi (kd_\mathrm{w})^2}{n^2}\left(\frac{ka}{2} - \frac{1}{24\,ka}\right) \tag{47}$$

satisfies

$$|r_n^\mathrm{A}| \leqslant \frac{M_\mathrm{A}}{n^3}\,. \tag{48}$$

For any $p \in \mathbb{N}$ and any $\psi \in [0, 2\pi)$, we define the series

$$\mathcal{J}_p'(\psi) = -\pi k d_\mathrm{w} \sum_{n=p+1}^{\infty} \frac{\cos n\psi}{n}\,, \tag{49}$$

$$\mathcal{J}_p''(\psi) = \pi (kd_\mathrm{w})^2\left(\frac{ka}{2} - \frac{1}{24\,ka}\right)\sum_{n=p+1}^{\infty}\frac{\cos n\psi}{n^2}\,, \tag{50}$$

and

$$\mathcal{J}_p'''(\psi) = \sum_{n=p+1}^{\infty}\left[\frac{1}{A_n} + \frac{\pi k d_\mathrm{w}}{n} - \frac{\pi (kd_\mathrm{w})^2}{n^2}\left(\frac{ka}{2} - \frac{1}{24\,ka}\right)\right]\cos n\psi\,, \tag{51}$$

which satisfy

$$\sum_{n=p+1}^{\infty}\frac{\cos n\psi}{A_n} = \mathcal{J}_p'(\psi) + \mathcal{J}_p''(\psi) + \mathcal{J}_p'''(\psi)\,. \tag{52}$$

A comparison with the harmonic series shows that the series $\mathcal{J}_0'(0)$ and $\mathcal{J}_p'(0)$ do not converge. For any ψ lying in the open interval $(0, 2\pi)$, $\mathcal{J}_0'(\psi)$ is a Fourier series expansion of a known function [11, Eq. 1.14]:

$$\mathcal{J}_0'(\psi) = \pi k d_\mathrm{w} \ln\left|2\sin\frac{\psi}{2}\right|\,. \tag{53}$$

Therefore, $\mathcal{J}_p'(\psi)$ converges and

$$\mathcal{J}_p'(\psi) = \pi k d_\mathrm{w}\left[\ln\left|2\sin\frac{\psi}{2}\right| + \sum_{n=1}^{p}\frac{\cos n\psi}{n}\right]. \tag{54}$$

Copyright © 2026 by Excem

For any ψ lying in the interval $[0, 2\pi)$, a comparison with the convergent series $\Sigma_{n=1}^{\infty} n^{-2}$ shows that $\mathcal{J}_0''(\psi)$ converges absolutely. Also, $\mathcal{J}_0''(\psi)$ is a Fourier series expansion of a known function [11, Eq. 1.27]:

$$\mathcal{J}_0''(\psi) = \pi (k d_\mathrm{w})^2 \left(\frac{ka}{2} - \frac{1}{24\,ka} \right) \frac{3\psi^2 - 6\pi\psi + 2\pi^2}{12} . \tag{55}$$

Therefore, $\mathcal{J}_p''(\psi)$ converges and

$$\mathcal{J}_p''(\psi) = \pi (k d_\mathrm{w})^2 \left(\frac{ka}{2} - \frac{1}{24\,ka} \right) \times \left(\frac{\pi^2}{6} - \frac{\pi\psi}{2} + \frac{\psi^2}{4} - \sum_{n=1}^{p} \frac{\cos n\psi}{n^2} \right) . \tag{56}$$

If $p \geqslant N_A$, it follows from (47) that

$$\mathcal{J}_p'''(\psi) = \sum_{n=p+1}^{\infty} r_n^\mathrm{A} \cos n\psi \, . \tag{57}$$

Using (48), (57) and a comparison with the convergent series $\Sigma_{n=1}^{\infty} n^{-3}$, we find that $\mathcal{J}_p'''(\psi)$ converges absolutely for any $p \in \mathbb{N}$, and that, if $p \geqslant N_\mathrm{A}$, then

$$|\mathcal{J}_p'''(\psi)| < M_\mathrm{A} \left(\zeta(3) - \sum_{n=1}^{p} \frac{1}{n^3} \right) , \tag{58}$$

where ζ denotes the Riemann zeta function [8, Ch. 23], and $\zeta(3) \simeq 1.20206$.

Using (52) allows us to assert that: the series in (2) does not converge at $\psi = 0$ and $\psi = 2\pi$, and that this series converges for $\psi \in (0, 2\pi)$, where we get

$$i_\mathrm{W}(\psi) = i_\mathrm{E}(\psi) + \frac{2V_0}{j\pi\eta} \left[\mathcal{J}_N'(\psi) + \mathcal{J}_N''(\psi) + \mathcal{J}_N'''(\psi) \right] , \tag{59}$$

in which N is the positive integer used in (3) to define $i_\mathrm{E}(\psi)$.

If tight values of N_A and M_A were known, (58) could be used to select the integer N in such a way that $N \geqslant N_\mathrm{A}$ and $|\mathcal{J}_N'''(\psi)|$ is small enough to ignore $\mathcal{J}_N'''(\psi)$ in the right-hand side of (59). Instead of this rigorous approach, it is simpler to use the following procedure:

- plot the absolute values of the complex frequency-dependent capacitances $|\mathcal{C}_\mathrm{R}/A_n|$ as a function of the frequency for several values of the positive integer n, as in Fig. 5;
- determine the minimum value of n_MIN of n above which the variations of $|\mathcal{C}_\mathrm{R}/A_n|$ as a function of the frequency may be ignored in the frequency range of interest, that is to say the minimum value of n above which the approximation (27) is valid;
- choose an integer $N_\mathrm{B} \geqslant 20$ such that $N_\mathrm{B} \geqslant n_\mathrm{MIN}$ and $N_\mathrm{B} \geqslant 1 + 0.9\, a/d_\mathrm{w}$, taking into account the fact that the greater the chosen N_B the more accurate the estimated $i_\mathrm{W}(\psi)$; and
- posit $N = N_\mathrm{B}$, this value being suitable for estimating $i_\mathrm{W}(\psi)$ using

$$i_\mathrm{W}(\psi) \simeq i_\mathrm{E}(\psi) + \frac{2V_0}{j\pi\eta} \mathcal{J}_N'(\psi) \, . \tag{60}$$

To elucidate why this procedure works, we assume that $t = d_\mathrm{w}/(2a) < 1/3$, and first note that we can use $N_\mathrm{B} \geqslant 20$ and the results of Section IV.B to show that, for any integer $q \geqslant N_\mathrm{B} - 1$, we have $r_q(t) < 0.0017$. Thus, for any integer $n \geqslant N_\mathrm{B}$, we can ignore Ξ_n, Ξ_{n+1} and Ξ_{n-1} in (7) to obtain

$$A_n = -\varsigma_n^{(1)} + \varsigma_n^{(2)} - \varsigma_n^{(3)} + \mathrm{S}_n(ka) \, , \tag{61}$$

where:

$$\varsigma_n^{(1)} \simeq \frac{n^2 P(nt)}{\pi ka} \tag{62}$$

with an absolute value of the relative error less than 0.2%;

$$\varsigma_n^{(2)} \simeq ka \, \frac{P((n+1)t) + P((n-1)t)}{2\pi} \tag{63}$$

with an absolute value of the relative error less than 0.2%; and, using (37),

$$\varsigma_n^{(3)} = ka\, n^2 \chi_{2\,n} = \frac{ka}{2\pi} \, \frac{4n^2}{(2n-1)(2n+1)} \, . \tag{64}$$

We find that, for a real number $x \geqslant 0.45$, the absolute value of the relative error of $P_\mathrm{L}(x)$ as a large-value approximation of $P(x)$ is less than 6.3%, in line with Fig. 7. Since, according to the procedure, $N_\mathrm{B} \geqslant 1 + 0.45/t$, we can assert that, for any integer $n \geqslant N_\mathrm{B}$, the absolute values of the relative errors in the approximations $P_\mathrm{L}(nt) \simeq P(nt)$, $P_\mathrm{L}((n+1)t) \simeq P((n+1)t)$ and $P_\mathrm{L}((n-1)t) \simeq P((n-1)t)$ are less than 6.3%. Since $N_\mathrm{B} \geqslant 20$ and $ka \leqslant 2.5$, it follows that, for any integer $n \geqslant N_\mathrm{B}$, we have

$$\frac{\varsigma_n^{(2)}}{\varsigma_n^{(1)}} \simeq \frac{(ka)^2}{n^2} \leqslant 1.6\% \tag{65}$$

and

$$\frac{\varsigma_n^{(3)}}{\varsigma_n^{(1)}} \simeq \frac{(ka)^2 t}{n} \leqslant 10.5\% \, , \tag{66}$$

hence the series resonance of $\mathfrak{L}_n$ and $\mathfrak{C}_n$, which by (25) corresponds to $-\varsigma_n^{(1)} + \varsigma_n^{(2)} - \varsigma_n^{(3)} = 0$, is not approached.

Since, according to the procedure, $N_\mathrm{B} \geqslant n_\mathrm{MIN}$ and n_MIN is such that, for any integer $n \geqslant n_\mathrm{MIN}$, the variations of $|\mathcal{C}_\mathrm{R}/A_n|$ as a function of the frequency can be ignored in the frequency range of interest, we can assert that, for any integer $n \geqslant N_\mathrm{B}$, in this frequency range, $\mathrm{S}_n(ka)$ can be ignored in the right-hand side of (61), hence

$$\frac{1}{A_n} \simeq -\frac{1}{\varsigma_n^{(1)}} \simeq -\frac{\pi k d_\mathrm{w}}{n} \, . \tag{67}$$

This validates the procedure, because the right-hand side of (67) is the first term of the right-hand side of (46), on which (60) is based. Note that, for the reason explained at the end of Section V, it also possible to use

$$i_\mathrm{W}(\psi) \simeq i_\mathrm{E}(\psi) + \frac{2V_0}{j\pi\eta} \left[\mathcal{J}_N'(\psi) + \mathcal{J}_N''(\psi) \right] \tag{68}$$

if $N = N_\mathrm{B} \geqslant 20 \times a/d_\mathrm{w}$.

We have shown that the procedure defined above to compute the current distribution $i_\mathrm{W}(\psi)$ in the loop antenna using (60) or (68) obviates: the convergence problems of the series in (2) proposed by Wu for the delta-gap source; and the inaccuracy of (3) close to the gap.

VII. EXAMPLE — PART 2

For the example of Section III, we have $1 + 0.9\, a/d_\mathrm{w} = 19$, and the curves of Fig. 5 suggest that $n_\mathrm{MIN} = 20$ should be appropriate for frequencies up to $f_\max$. More precisely, Fig. 9 shows the ratio $[M_n(\Delta_\mathrm{f}) - m_n(\Delta_\mathrm{f})]/m_n(\Delta_\mathrm{f})$ as a function of the positive integer n, where $M_n(\Delta_\mathrm{f})$ is the maximum value of $|\mathcal{C}_\mathrm{R}/A_n|$ computed in a frequency interval Δ_f, and $m_n(\Delta_\mathrm{f})$ is the minimum value of $|\mathcal{C}_\mathrm{R}/A_n|$ computed in Δ_f. In Fig. 9, for a frequency range of interest Δ_f extending up to $f_\max \simeq 426\,\mathrm{MHz}$, we see that this ratio is less than 0.8% for $n = 20$, hence the value $n_\mathrm{MIN} = 20$ is appropriate and N_B can be any integer greater than or equal to 20.

According to the procedure of Section VI, an accurate current distribution for the delta-gap source is $i_\mathrm{W}(\psi) \simeq i_\mathrm{A}(\psi)$, where $i_\mathrm{A}(\psi)$ is the right-hand side of (60), that is to say

$$\begin{aligned} i_\mathrm{A}(\psi) = i_\mathrm{E}(\psi) + \frac{2V_0}{j\pi\eta}\mathcal{J}'_N(\psi) = \\ \frac{V_0}{j\pi\eta}\left\{\frac{1}{A_0} + 2\pi k d_\mathrm{w} \ln\left|2\sin\frac{\psi}{2}\right|\right. \\ \left. + 2\sum_{n=1}^{N}\left(\frac{1}{A_n} + \frac{\pi k d_\mathrm{w}}{n}\right)\cos n\psi\right\}, \end{aligned} \tag{69}$$

in which $N = N_\mathrm{B}$. For $V_0 = 1$ Volt, we get

$$\lim_{\psi\to 0} i_\mathrm{A}(\psi) = \lim_{\psi\to 2\pi} i_\mathrm{A}(\psi) = j\infty\,. \tag{70}$$

Fig. 10 to Fig. 12 show $|i_\mathrm{E}(\psi)|$ and $|i_\mathrm{A}(\psi)|$, computed for $N = 20$ at a frequency close to $f_\max$. Fig. 13 to Fig. 15 show $|i_\mathrm{E}(\psi)|$ and $|i_\mathrm{A}(\psi)|$, computed for $N = 40$ at this frequency. Ripples depending on N are visible in the curves representing $|i_\mathrm{E}(\psi)|$ in Fig. 10, Fig. 12, Fig. 13 and Fig. 15. They are artifacts, because their shapes depend on N. The Gibbs phenomenon is the oscillatory behavior of the partial sums of a Fourier series of a piecewise continuously differentiable periodic function around discontinuities [12]. Said “ripples” visible in the curves representing $|i_\mathrm{E}(\psi)|$ are the result of a similar but different phenomenon: it is an oscillatory behavior of the partial sums of a Fourier series of a piecewise continuously differentiable periodic function near vertical asymptotes. The curves representing $|i_\mathrm{A}(\psi)|$ in Fig. 11, Fig. 12, Fig. 14 and Fig. 15 show that the procedure proposed in Section VI eliminates this unwanted phenomenon, and provides the expected behavior near the vertical asymptotes defined by (70).

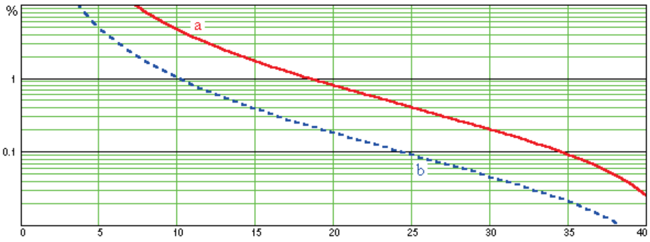

FIGURE 9. The ratio $[M_n(\Delta_\mathrm{f}) - m_n(\Delta_\mathrm{f})]/m_n(\Delta_\mathrm{f})$ defined in Section VII, versus n. Curve “a” is for the frequency interval $\Delta_\mathrm{f} \simeq [100\ \mathrm{kHz}, 427\ \mathrm{MHz}]$. Curve “b” is for the frequency interval $\Delta_\mathrm{f} \simeq [100\ \mathrm{kHz}, 204\ \mathrm{MHz}]$.

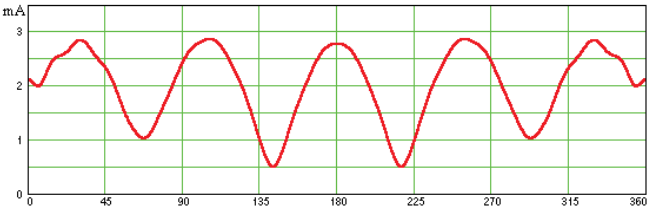

FIGURE 10. $|i_\mathrm{E}(\psi)|$ versus ψ in degrees in the interval [0.0°, 360.0°], at about $426.58\ \mathrm{MHz}$, computed for $N = 20$ and $V_0 = 1$ V.

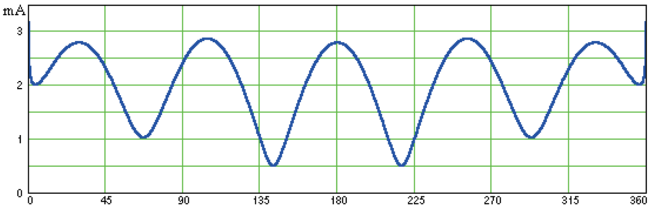

FIGURE 11. $|i_\mathrm{A}(\psi)|$ versus ψ in degrees in the interval [0.1°, 359.9°], at about $426.58\ \mathrm{MHz}$, computed for $N = 20$ and $V_0 = 1$ V.

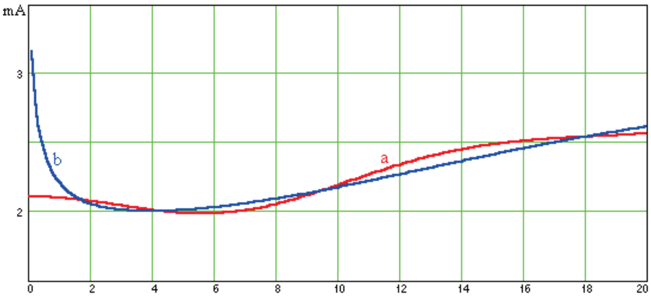

FIGURE 12. $|i_\mathrm{E}(\psi)|$ and $|i_\mathrm{A}(\psi)|$ at about $426.58\ \mathrm{MHz}$, for $N = 20$ and $V_0 = 1$ V. Curve “a” is $|i_\mathrm{E}(\psi)|$ versus ψ in degrees in the interval [0.0°, 20°]. Curve “b” is $|i_\mathrm{A}(\psi)|$ versus ψ in degrees in the interval [0.1°, 20°].

Fig. 16 shows that changing N from $N = 20$ to $N = 40$ has a very small effect on $|i_\mathrm{A}(\psi)|$. This is consistent with the fact that the procedure proposed in Section VI provides accurate values of $i_\mathrm{W}(\psi)$ at frequencies up to $f_\max$, using values of N as small as $N = 20$ or $N = 40$, hence at a low computational cost.

VIII. IMPEDANCE AND TERMINAL-ZONE NETWORK

A. GENERAL RESULTS

By (24), A_0 corresponds to the impedance

$$z_0 = j\pi\eta A_0 = j\omega\,\mathfrak{L}_0 + j\pi\eta\,\mathrm{S}_0(ka)\,. \tag{71}$$

By (25), for any positive integer n, A_n corresponds to the impedance

$$z_n = \frac{j\pi\eta A_n}{2} = \frac{1}{j\omega\,\mathfrak{C}_n} + j\omega\,\mathfrak{L}_n + \frac{j\pi\eta}{2}\,\mathrm{S}_n(ka)\,. \tag{72}$$

Combined with (4), (71)–(72) suggest the possible role of $N+1$ inductors of inductances $\mathfrak{L}_0, \ldots, \mathfrak{L}_N$, respectively, and of N capacitors of capacitances $\mathfrak{C}_1, \ldots, \mathfrak{C}_N$, respectively, in an equivalent circuit of Y_δ. This path will not be explored further in this article. However, we can use (4) to write

$$Y_\delta = \sum_{n=0}^{N} \frac{1}{z_n}\,, \tag{73}$$

Copyright ©2026 by Excem

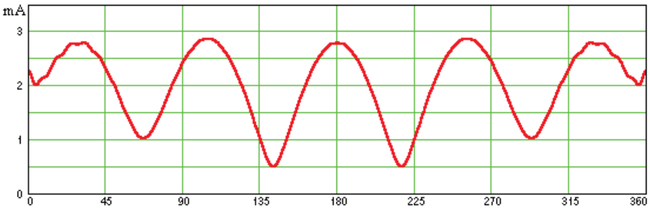

FIGURE 13. $|i_{\mathrm{E}}(\psi)|$ versus ψ in degrees in the interval [0.0°, 360.0°], at about $426.58\ \mathrm{MHz}$, computed for $N = 40$ and $V_0 = 1$ V.

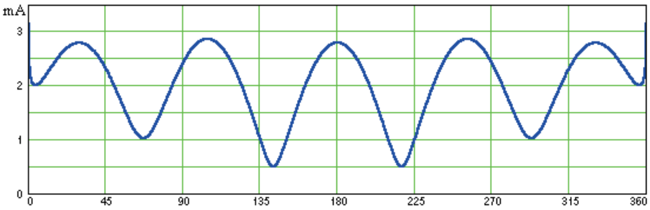

FIGURE 14. $|i_{\mathrm{A}}(\psi)|$ versus ψ in degrees in the interval [0.1°, 359.9°], at about $426.58\ \mathrm{MHz}$, computed for $N = 40$ and $V_0 = 1$ V.

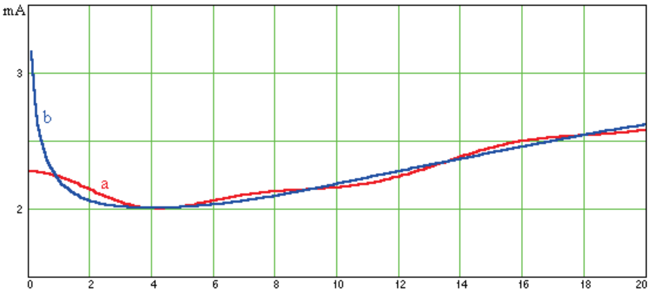

FIGURE 15. $|i_{\mathrm{E}}(\psi)|$ and $|i_{\mathrm{A}}(\psi)|$ at about $426.58\ \mathrm{MHz}$, for $N = 40$ and $V_0 = 1$ V. Curve “a” is $|i_{\mathrm{E}}(\psi)|$ versus ψ in degrees in the interval [0.0°, 20°]. Curve “b” is $|i_{\mathrm{A}}(\psi)|$ versus ψ in degrees in the interval [0.1°, 20°].

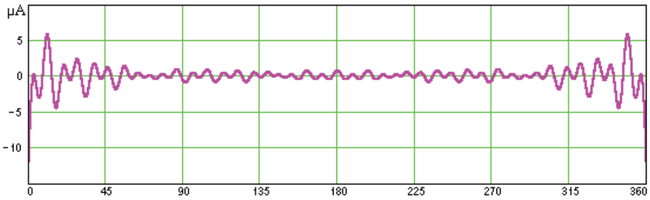

FIGURE 16. In the interval [0.1°, 359.9°], the difference $|i_{\mathrm{A}}(\psi)|$ shown in Fig 14 (for $N = 40$) minus $|i_{\mathrm{A}}(\psi)|$ shown in Fig 11 (for $N = 20$).

hence, in this article, Y_δ is exactly the admittance obtained by connecting $z_0, \ldots, z_N$ in parallel.

We defined n_{MIN} as the smallest positive integer such that, for any integer $n \geqslant n_{\mathrm{MIN}}$, the approximation (27) is valid in the frequency range of interest, or equivalently such that

$$z_n \simeq \frac{1}{j\omega\,\mathfrak{C}_n} \tag{74}$$

in the frequency range of interest. Thus, if $N > n_{\mathrm{MIN}}$, each term $1/z_n$ of index $n = n_{\mathrm{MIN}} + 1$ to $n = N$ in the right-hand side of (73) accurately corresponds, in the frequency range of interest, to the addition of $j\omega\,\mathfrak{C}_n$, or equivalently to the connection of a parallel capacitance $\mathfrak{C}_n$. If N becomes large, it follows from (41) that the sum of the added capacitances also becomes large (by [13, Sec. 1.2.2]).

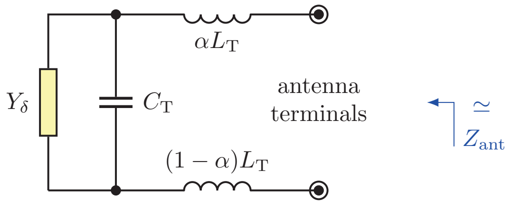

FIGURE 17. A terminal-zone network, connected to Y_δ.

The antenna impedance Z_{ant} is normally defined as the impedance seen, looking into the antenna, from a reference plane of a feeder. It depends on a, d_{w}, k, ψ_{min} shown in Fig. 1, and also on the actual configuration of the antenna and the feeder in the vicinity of the terminals of the loop antenna.

As said in the introduction, models of Z_{ant} traditionally includes a terminal-zone network, used to convert the admittance Y_δ, which only depends on a, d_{w} and k, into an impedance close to Z_{ant}. The terminal-zone network used in [1] is shown in Fig. 17, where α is an arbitrary real number. This terminal-zone network comprises a lumped inductance L_{T} and a lumped capacitance C_{T}, both of which need not be nonnegative, and is such that

$$Z_{\mathrm{ant}} \simeq \frac{1}{Y_\delta + j\omega C_{\mathrm{T}}} + j\omega L_{\mathrm{T}}\,. \tag{75}$$

The facts that each term $1/z_n$ of index $n > n_{\mathrm{MIN}}$ accurately corresponds, in the frequency range of interest, to the connection of a parallel capacitance $\mathfrak{C}_n$, and that, if N becomes large, the sum of the added capacitances also becomes large, are not a problem in the context of (75), for the following reason. Let us for instance assume that, for a value $N' \geqslant n_{\mathrm{MIN}}$ of the integer N, the values C'_{T} of C_{T} and L'_{T} of L_{T} provide an optimal model in a frequency range of interest. It follows from the previous discussion that, for a value $N'' > N'$ of the integer N, the values

$$C''_{\mathrm{T}} = C'_{\mathrm{T}} - \sum_{n=N'+1}^{N''} \mathfrak{C}_n \tag{76}$$

of C_{T} and $L''_{\mathrm{T}} = L'_{\mathrm{T}}$ of L_{T} provide almost exactly the same optimal model in the frequency range of interest.

Accordingly, we can assert that: in the frequency range of interest, the model defined by Fig. 17 or by the right-hand side of (75) gives almost the same value of the antenna impedance for any $N \geqslant n_{\mathrm{MIN}}$, if C_{T} is well be chosen as a function of N. More generally, this statement also applies to any terminal-zone network comprising an arbitrary capacitor C_{T} connected in parallel with Y_δ.

Note that this statement invalidate a reasoning proposed in the section “Evaluation of the Graphs” of [4], the section “Evaluation of the Tables” of [5], and in [7, Sec. 11.4], according to which the fact that the imaginary part of the computed Y_δ varies when N changes would be a symptom of a poor accuracy. In contrast, we have established that, if $N \geqslant n_{\mathrm{MIN}}$, this phenomenon is almost exactly compensated for by a proper terminal-zone network.

Copyright ©2026 by Excem

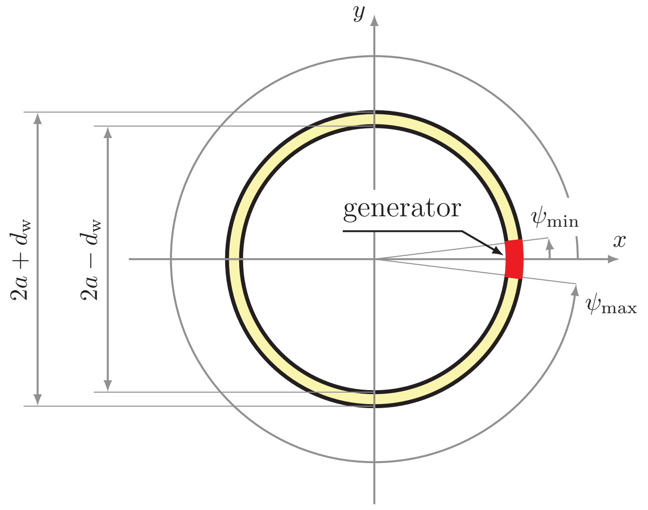

FIGURE 18. The second theoretical loop antenna, and, shown in red, the generator containing the delta-gap source.

B. EXAMPLE

We now refer to the loop antenna excited by a delta-gap source of voltage V_0 located at $\psi = 0$ as the "first theoretical loop antenna". For any $\psi_{\rm min} \ll \pi/18$, we define a "second theoretical loop antenna" as the part of the first theoretical loop antenna lying in the interval $[\psi_{\rm min}, \psi_{\rm max}]$, in which $\psi_{\rm max} = 2\pi - \psi_{\rm min}$ as usual, and we define a generator as the delta-gap source and the part of the first theoretical loop antenna that does not belong to the second theoretical loop antenna. This configuration is shown in Fig 18.

We said above that $Z_{\rm ant}$ is normally defined as the impedance seen, looking into the antenna, from a reference plane of a feeder, but this is not possible in this configuration that does not include a feeder. To define the voltage between the terminals of the second theoretical loop antenna, we can use an integration path that is an arc of the circle of radius $a - d_{\rm w}/2$, lying in the plane $z = 0$, and having its center at $x = y = z = 0$. In this case, this voltage is V_0. It follows that $Z_{\rm ant} = V_0 / i_{\rm W}(\psi_{\rm min})$.

If N is determined according the procedure of Section VI, we get $Z_{\rm ant} \simeq 1/Y_\Delta(\psi_{\rm min})$, where

$$Y_\Delta(\psi_{\rm min}) = \frac{i_{\rm E}(\psi_{\rm min})}{V_0} + \frac{2}{j\pi\eta}\mathcal{J}'_N(\psi_{\rm min}) = \frac{1}{j\pi\eta}\left\{\frac{1}{A_0} + 2\pi k d_{\rm w} \ln\left|2\sin\frac{\psi_{\rm min}}{2}\right| + 2\sum_{n=1}^{N}\left(\frac{1}{A_n} + \frac{\pi k d_{\rm w}}{n}\right)\cos n\psi_{\rm min}\right\}, \quad (77)$$

in which $N = N_{\rm B}$. Thus, if $N\psi_{\rm min} \ll 1$, we find that (75) can be used with the parameters $L_{\rm T} = 0$ nH and

$$C_{\rm T} = -2\epsilon d_{\rm w}\left\{\ln\left|2\sin\frac{\psi_{\rm min}}{2}\right| + \sum_{n=1}^{N}\frac{\cos n\psi_{\rm min}}{n}\right\}. \quad (78)$$

Fig 19 shows the impedance $Y_\Delta(\psi_{\rm min})$ computed using the same parameters as in Section III and Section VII, for different values of $\psi_{\rm min}$.

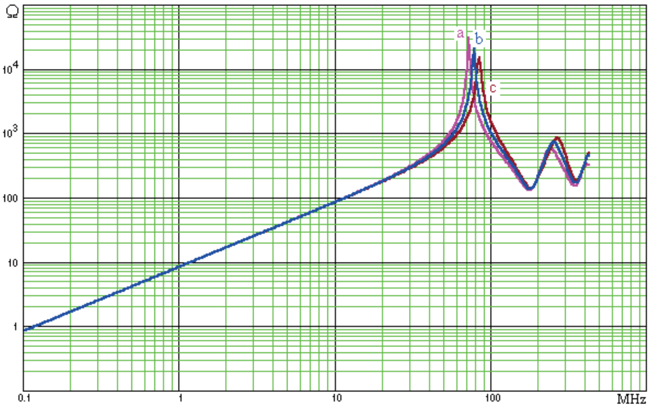

FIGURE 19. Impedance $1/Y_\Delta(\psi_{\rm min})$ versus frequency up to about 426.58 MHz, for $N = 20$. Curve "a" is for $\psi_{\rm min} = 0.1°$. Curve "b" is for $\psi_{\rm min} = 1.0°$. Curve "c" corresponds to $\psi_{\rm min} = 6.0°$.

At a frequency f, for $L_{\rm T} = 0$ nH, a specified value of $C_{\rm T}$, and $N = 20$, let $\rho_{\rm Y}(f, C_{\rm T})$ be the absolute value of the relative error of the admittance given by the model of Fig. 17, compared to $Y_\Delta(\psi_{\rm min})$ regarded as an exact value. If $\psi_{\rm min} = 0.1°$, (78) gives $C_{\rm T} \simeq 682.614$ fF, and, over the frequency interval $\Delta_{\rm f} \simeq [100\ {\rm kHz}, 427\ {\rm MHz}]$, we find that: if $C_{\rm T} = 682.510$ fF, then $\rho_{\rm Y}(f, C_{\rm T}) < 54 \times 10^{-6}$. If $\psi_{\rm min} = 1.0°$, (78) gives $C_{\rm T} \simeq 119.574$ fF, and, over the frequency interval $\Delta_{\rm f}$, we find that: if $C_{\rm T} = 109.710$ fF, then $\rho_{\rm Y}(f, C_{\rm T}) < 615 \times 10^{-6}$. If $\psi_{\rm min} = 6.0°$, (78) gives a useless result, and, over the frequency interval $\Delta_{\rm f}$, we find that: if $C_{\rm T} = -399$ fF, then $\rho_{\rm Y}(f, C_{\rm T}) < 0.023$.

IX. CONCLUSION

We have studied a lossless single-turn circular wire loop antenna lying in a homogeneous and lossless medium. The main new results obtained is this article are:

- a detailed analysis of the Wu-King factors, which establishes that they can be accurately estimated at frequencies far greater than $f_{\rm max}$;
- a mathematical explanation of the convergence of the series in (2) for $\psi \in (0, 2\pi)$, and of its lack of convergence at $\psi = 0$ and $\psi = 2\pi$;
- a procedure and the explicit formulas (60), (68) and (69) for computing the distribution of the current if the loop antenna is excited by a delta-gap source at $f \leqslant f_{\rm max}$, which obviate the convergence problems of (2) and the inaccuracy of (3) close to gap; and
- a proof of the fact that a model of the loop antenna impedance up to $f_{\rm max}$, this model comprising a suitable terminal-zone network, provides an impedance that practically does not depend on the number N of terms used in (3), if N exceeds a suitable minimum value, and if a capacitance used in the terminal-zone network is correctly determined as a function of N.

Combined with the results of [1] and [14], this work can be used to obtain improved models of loop antennas and shielded loop antennas used to produce known electromagnetic fields or to measure electromagnetic fields. It is therefore useful for electromagnetic compatibility.

APPENDIX A

In this Appendix A, we study the coefficients $\chi_{s\,r}$ defined by (8)–(9), where r and s lie in $\mathbb{N}$ and $s \geqslant 1$.

By [8, Sec. 6.1.17], we have

$$\Gamma\Big(\frac{s}{2} - r + 0.5\Big)\,\Gamma\Big(-\frac{s}{2} + r + 0.5\Big) = \frac{\pi}{\sin\left(\pi\left[\dfrac{s}{2} - r + 0.5\right]\right)}\,. \tag{79}$$

hence, if s is even, then

$$\Gamma\Big(\frac{s}{2} - r + 0.5\Big) = \frac{\pi(-1)^{\frac{s}{2}-r}}{\Gamma\Big(-\dfrac{s}{2} + r + 0.5\Big)}\,. \tag{80}$$

If s is even, we use [8, Sec. 6.1.12] to get

$$\Gamma\Big(\frac{s}{2} + r + 0.5\Big) = \frac{1\cdot 3\cdots(s+2r-1)}{2^{r+s/2}}\,\Gamma\Big(\frac{1}{2}\Big)\,, \tag{81}$$

hence

$$\Gamma\Big(\frac{s}{2} + r + 0.5\Big) = \frac{(s+2r)!}{2^{2r+s}\,(r+s/2)!}\,\Gamma\Big(\frac{1}{2}\Big)\,. \tag{82}$$

If s is even and $s \leqslant 2r-2$, then $r - s/2 \geqslant 1$, hence we can use [8, Sec. 6.1.12] to obtain

$$\Gamma\Big(-\frac{s}{2} + r + 0.5\Big) = \frac{1\cdot 3\cdots(2r-s-1)}{2^{r-s/2}}\,\Gamma\Big(\frac{1}{2}\Big)\,, \tag{83}$$

hence

$$\Gamma\Big(-\frac{s}{2} + r + 0.5\Big) = \frac{(2r-s)!}{2^{2r-s}\,(r-s/2)!}\,\Gamma\Big(\frac{1}{2}\Big)\,. \tag{84}$$

If s is even and $s \leqslant 2r-2$, using (84) in (80), we get

$$\Gamma\Big(\frac{s}{2} - r + 0.5\Big) = \frac{\pi(-1)^{\frac{s}{2}-r}\,2^{2r-s}\,(r-s/2)!}{(2r-s)!\,\Gamma\Big(\dfrac{1}{2}\Big)}\,. \tag{85}$$

If s is even and $s \leqslant 2r-2$, using (82) and (85), we obtain

$$\Gamma\Big(\frac{s}{2} + r + 0.5\Big)\,\Gamma\Big(\frac{s}{2} - r + 0.5\Big) = \frac{\pi(-1)^{\frac{s}{2}-r}\,(2r+s)!\,(r-s/2)!}{2^{2s}(2r-s)!\,(r+s/2)!}\,. \tag{86}$$

These considerations allow us to clarify the definition of $\chi_{s\,r}$ in (8)–(9), as follows:

- if s is even and $s \leqslant 2r-1$, it follows from (86), that
$$\chi_{s\,r} = \frac{2^{2s}(2r-s)!\,(r+s/2)!}{s\,\pi\,(2r+s)!\,(r-s/2)!}\,, \tag{87}$$
which is a positive real number;
- if s is odd and $s \leqslant 2r-1$, then
$$\chi_{s\,r} = 0\,; \tag{88}$$
- if $s = 2r$, it follows from (82) and [8, Sec. 6.1.8], that
$$\chi_{2r\,r} = \frac{2^{4r-1}\,(2r)!}{r\,(4r)!\,\pi}\,, \tag{89}$$
which is a positive real number;
- if s is even and $s \geqslant 2r$, then $\chi_{s\,r}$ is the real number given by (8), which satisfies
$$\chi_{s\,r} = (-1)^{\frac{s}{2}+r}\,|\chi_{s\,r}|\,, \tag{90}$$
because $\Gamma(x)$ is real and positive for any positive $x \in \mathbb{R}$; and
- if s is odd and $s \geqslant 2r$, then $\chi_{s\,r}$ is the imaginary number given by (9), which satisfies
$$\chi_{s\,r} = j(-1)^{\frac{s+1}{2}+r}\,|\chi_{s\,r}|\,. \tag{91}$$

Going back to the case where s is even and $s \leqslant 2r-1$, if follows from (87) that

$$\chi_{s\,r} = \frac{2^{2s}\displaystyle\prod_{i=r+1-s/2}^{r+s/2} i}{s\,\pi\displaystyle\prod_{i=2r-s+1}^{2r+s} i}\,, \tag{92}$$

which leads us to

$$\chi_{s\,r} = \frac{2^{s}}{s\,\pi\displaystyle\prod_{i=r-s/2}^{r-1+s/2}(2i+1)}\,. \tag{93}$$

APPENDIX B

In this Appendix B, n being a nonnegative integer, we determine the radius of convergence of the power series $\mathrm{S}_n(ka)$ defined by (13).

Using [8, Eq. 6.1.6] in (9), and Stirling's formula [8, Eq. 6.1.37] in (8)–(9), we find that, for a fixed $r \in \mathbb{N}$, if $s \geqslant 2r+1$ and s becomes large, then $\chi_{s\,r}$ satisfies

$$|\chi_{s\,r}| = \frac{e^{s+1}\left(\dfrac{s}{2}+r+0.5\right)^{-(\frac{s}{2}+r)}\left(\dfrac{s}{2}-r+0.5\right)^{-(\frac{s}{2}-r)}}{2\pi s\left(1+O\Big(\dfrac{1}{s}\Big)\right)}\,, \tag{94}$$

where we have used Landau's big-O notation. Thus, we get

$$|\chi_{s\,r}| \leqslant \frac{e^{s+1}\left(\dfrac{s}{2}-r+0.5\right)^{-(\frac{s}{2}+r)}\left(\dfrac{s}{2}-r+0.5\right)^{-(\frac{s}{2}-r)}}{2\pi s\left(1+O\Big(\dfrac{1}{s}\Big)\right)}\,, \tag{95}$$

hence

$$|\chi_{s\,r}| \leqslant \frac{e^{s+1}}{2\pi s\left(\dfrac{s}{2}-r+0.5\right)^{s}}\left(1+O\Big(\frac{1}{s}\Big)\right)\,. \tag{96}$$

Using (96) in (11), we find

$$\lim_{q\to\infty}|\alpha_{q\,0}|^{1/q} = \lim_{q\to\infty}|\chi_{(q-1)\,1}|^{1/q} = 0\,. \tag{97}$$

By (12), for any integer $q \geqslant 2$ and any positive integer n, we have

$$|\alpha_{q\,n}| \leqslant \frac{|\chi_{(q-1)\,(n-1)}| + |\chi_{(q-1)\,(n+1)}|}{2} + n^2|\chi_{(q+1)\,n}|\,. \tag{98}$$

Copyright ©2026 by Excem

Among $|\chi_{(q-1)\,(n-1)}|$, $|\chi_{(q-1)\,(n+1)}|$ and $|\chi_{(q+1)\,n}|$, we find that $|\chi_{(q-1)\,(n+1)}|$ produces the largest right-hand side in (96) when q becomes large. Thus, using (96) in (98), we get, when q becomes large,

$$|\alpha_{q\,n}| \leqslant \frac{e^q\,(1+n^2)}{2\pi(q-1)\left(\dfrac{q-1}{2}-n-0.5\right)^{q-1}}\left(1+O\left(\frac{1}{q}\right)\right). \tag{99}$$

Hence, for any positive integer n,

$$\lim_{q\to\infty}|\alpha_{q\,n}|^{1/q}=0\,. \tag{100}$$

It follows from (97) and (100) that, for any $n\in\mathbb{N}$, the radius of convergence of the power series $\mathrm{S}_n(ka)$ defined by (13) is infinity.

APPENDIX C

In this Appendix C, n being a fixed nonnegative integer, we study how fast the power series $\mathrm{S}_n(ka)$ defined by (13) converges, to assess the accuracy of its partial sums. More precisely, for a specified $d_{\mathrm{A}}(n)$, we want to obtain an upper bound on the absolute error $\mathcal{E}_n(ka)$ defined by (15).

For any $n\in\mathbb{N}$, we define

$$\mathrm{s}_n^{\mathrm{poly}}(ka)=\sum_{q=2}^{2n+2}\alpha_{q\,n}(ka)^q\,, \tag{101}$$

$$\mathrm{s}_n^{\mathrm{odd}}(ka)=\sum_{p=n+1}^{\infty}\alpha_{(2p+1)\,n}(ka)^{2p+1} \tag{102}$$

and

$$\mathrm{s}_n^{\mathrm{even}}(ka)=\sum_{p=n+2}^{\infty}\alpha_{(2p)\,n}(ka)^{2p}\,. \tag{103}$$

It follows from (97) and (100) that, for any $n\in\mathbb{N}$, the radii of convergence of the power series $\mathrm{s}_n^{\mathrm{odd}}(ka)$ and $\mathrm{s}_n^{\mathrm{even}}(ka)$ are infinity. Thus, using [13, Sec. 1.1.2.3], we get

$$\mathrm{S}_n(ka)=\mathrm{s}_n^{\mathrm{poly}}(ka)+\mathrm{s}_n^{\mathrm{odd}}(ka)+\mathrm{s}_n^{\mathrm{even}}(ka)\,. \tag{104}$$

The polynomial $\mathrm{s}_n^{\mathrm{poly}}(ka)$ will be used in Appendix H. Since we are investigating the convergence of $\mathrm{S}_n(ka)$, we will now examine the series $\mathrm{s}_n^{\mathrm{odd}}(ka)$ and $\mathrm{s}_n^{\mathrm{even}}(ka)$. For this purpose, it is convenient to define: the real coefficients $\beta_{(2p+1)\,n}=\alpha_{(2p+1)\,n}$, where p is a positive integer such that $p\geqslant n+1$; the real coefficients $\beta_{(2p)\,n}=j\alpha_{(2p)\,n}$, where p is a positive integer such that $p\geqslant n+2$; and the power series

$$\mathrm{s}_n^{\mathrm{Odd}}(x)=\sum_{p=n+1}^{\infty}\beta_{(2p+1)\,n}\;x^p \tag{105}$$

and

$$\mathrm{s}_n^{\mathrm{Even}}(x)=\sum_{p=n+2}^{\infty}\beta_{(2p)\,n}\;x^p\,, \tag{106}$$

where $x\in\mathbb{R}$ and $x\geqslant 0$. Using (102)–(103) and (105)–(106), we find that the radii of convergence of the power series $\mathrm{s}_n^{\mathrm{Odd}}(x)$ and $\mathrm{s}_n^{\mathrm{Even}}(x)$ are infinity,

$$\mathrm{s}_n^{\mathrm{odd}}(ka)=ka\;\mathrm{s}_n^{\mathrm{Odd}}((ka)^2) \tag{107}$$

and

$$\mathrm{s}_n^{\mathrm{even}}(ka)=-j\;\mathrm{s}_n^{\mathrm{Even}}((ka)^2)\,. \tag{108}$$

For a specified integer $d_{\mathrm{A}}(n)$ such that $d_{\mathrm{A}}(n)\geqslant 2n+3$, it follows from (104)–(108) that

$$\begin{aligned}\sum_{q=d_{\mathrm{A}}(n)+1}^{\infty}\alpha_{q\,n}(ka)^q&=-j\sum_{2p>d_{\mathrm{A}}(n)}^{\infty}\beta_{(2p)\,n}(ka)^{2p}\\&\quad+ka\sum_{2p+1>d_{\mathrm{A}}(n)}^{\infty}\beta_{(2p+1)\,n}(ka)^{2p}\,,\end{aligned} \tag{109}$$

where both series of the right-hand side are real. Thus,

$$\begin{aligned}\left|\sum_{q=d_{\mathrm{A}}(n)+1}^{\infty}\alpha_{q\,n}(ka)^q\right|^2&\leqslant\left|\sum_{2p>d_{\mathrm{A}}(n)}^{\infty}\beta_{(2p)\,n}(ka)^{2p}\right|^2\\&\quad+|ka|^2\left|\sum_{2p+1>d_{\mathrm{A}}(n)}^{\infty}\beta_{(2p+1)\,n}(ka)^{2p}\right|^2\end{aligned} \tag{110}$$

In the case $n=0$, by (8), (9) and (11) we have, for any integer $q\geqslant 3$,

$$\beta_{q\,0}=\frac{\epsilon_{q\,0}}{(q-1)\,\Gamma\left(\dfrac{q}{2}+1\right)\Gamma\left(\dfrac{q}{2}-1\right)} \tag{111}$$

where

$$\epsilon_{q\,0}=\begin{cases}(-1)^{\frac{q+1}{2}} & \text{if } q \text{ is odd}\\(-1)^{\frac{q}{2}} & \text{if } q \text{ is even}\end{cases}. \tag{112}$$

Accordingly, for any $q\geqslant 3$, the denominator of (111) being positive, we have

$$\mathrm{sgn}\,(\beta_{q\,0})=\mathrm{sgn}\,(\epsilon_{q\,0})\,, \tag{113}$$

where sgn is the signum function, which, for an arbitrary real number y, is such that $\mathrm{sgn}(y)$ is equal to 1 if y is positive, 0 if $y=0$ or -1 if y is negative. Note that (11), (90) and (91) also lead to (113). The results (111) and (113) allow us to assert that, for all positive $x\in\mathbb{R}$:

- the terms $\beta_{(2p+1)\,0}\;x^p$ of the series $\mathrm{s}_0^{\mathrm{Odd}}(x)$ tend to 0 as $p\to\infty$;
- the terms $\beta_{(2p)\,0}\;x^p$ of the series $\mathrm{s}_0^{\mathrm{Even}}(x)$ tend to 0 as $p\to\infty$; and
- the series $\mathrm{s}_0^{\mathrm{Odd}}(x)$ and $\mathrm{s}_0^{\mathrm{Even}}(x)$ are alternating.

We now study the sign of $|\beta_{q\,0}|-|\beta_{(q+2)\,0}\;x|$ for $x>0$ and $q\geqslant 3$. By (111) and [8, Sec. 6.1.15], we have

$$\begin{aligned}|\beta_{q\,0}|-|\beta_{(q+2)\,0}\;x|&=\frac{1}{D_{q\,0}}\\&\times\left((q+1)\,\frac{q}{2}\left(\frac{q}{2}-1\right)\left(\frac{q}{2}+1\right)-x(q-1)\,\frac{q}{2}\right),\end{aligned} \tag{114}$$

where

$$D_{q\,0}=(q-1)(q+1)\Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+2\right). \tag{115}$$

For a positive $x\in\mathbb{R}$, we define the integer

$$P_0^{\min}(x)=\mathrm{ceil}\left(\sqrt{x+1}\right) \tag{116}$$

Any term of degree $p \geqslant P_0^{\min}(x)$ of $\mathrm{s}_0^{\mathrm{Odd}}(x)$ or $\mathrm{s}_0^{\mathrm{Even}}(x)$ has a coefficient $\beta_{q\,0}$ such that $q \geqslant 2P_0^{\min}(x)$, hence $q \geqslant 4$ and $q \geqslant 2\sqrt{x+1}$, hence

$$\left(\frac{q}{2}-1\right)\left(\frac{q}{2}+1\right) \geqslant x \tag{117}$$

and

$$|\beta_{q\,0}| - |\beta_{(q+2)\,0}\ x| > 0\,, \tag{118}$$

where we have used (114). Thus, $P_0^{\min}(x)$ is such that: the absolute values of the terms of degree $p \geqslant P_0^{\min}(x)$ of $\mathrm{s}_0^{\mathrm{Odd}}(x)$ form a decreasing sequence; and the absolute values of the terms of degree $p \geqslant P_0^{\min}(x)$ of $\mathrm{s}_0^{\mathrm{Even}}(x)$ form a decreasing sequence.

Thus, if $P_0'(x)$ and $P_0''(x)$ are two integers greater than or equal to $P_0^{\min}(x)$, we can use [13, Sec. 1.3.3] to assert that

$$\left|\sum_{p=P_0'(x)}^{\infty} \beta_{(2p)\,0}\ x^p\right| \leqslant \left|\beta_{(2P_0'(x))\,0}\ x^{P_0'(x)}\right| \tag{119}$$

and

$$\left|\sum_{p=P_0''(x)}^{\infty} \beta_{(2p+1)\,0}\ x^p\right| \leqslant \left|\beta_{(2P_0''(x)+1)\,0}\ x^{P_0''(x)}\right|. \tag{120}$$

In the right-hand side of (110), for $n = 0$, the first non-zero coefficient of one of the series is $\beta_{(d_{\mathrm{A}}(0)+1)\,0}$, and the first non-zero coefficient of the other series is $\beta_{(d_{\mathrm{A}}(0)+2)\,0}$. Thus, it follows from (110), (116) and (119)–(120) that, for a specified integer $d_{\mathrm{A}}(0)$ such that

$$d_{\mathrm{A}}(0) \geqslant 2\,\mathrm{ceil}\left(\sqrt{(ka)^2+1}\right), \tag{121}$$

we have

$$\left|\sum_{q=d_{\mathrm{A}}(0)+1}^{\infty} \alpha_{q\,0}(ka)^q\right| \leqslant \mathfrak{E}_0\left(ka, d_{\mathrm{A}}(0)\right)\,, \tag{122}$$

where, for any integer $q \geqslant 1$,

$$\mathfrak{E}_0(ka,q) = \sqrt{\begin{array}{l}\left[|\alpha_{(q+1)\,0}|\,(ka)^{q+1}\right]^2\\ \qquad +\left[|\alpha_{(q+2)\,0}|\,(ka)^{q+2}\right]^2\end{array}}\,. \tag{123}$$

In the case $n \geqslant 1$, by (8), (9) and (12) we have, for any integer $q \geqslant 2n+3$,

$$\begin{aligned}\beta_{q\,n} = \epsilon_{q\,n}\Bigg[&\frac{1}{2(q-1)\,\Gamma\left(\frac{q}{2}+n-1\right)\Gamma\left(\frac{q}{2}-n+1\right)}\\ &+\frac{1}{2(q-1)\,\Gamma\left(\frac{q}{2}+n+1\right)\Gamma\left(\frac{q}{2}-n-1\right)}\\ &-\frac{n^2}{(q+1)\,\Gamma\left(\frac{q}{2}+n+1\right)\Gamma\left(\frac{q}{2}-n+1\right)}\Bigg],\end{aligned} \tag{124}$$

where

$$\epsilon_{q\,n} = \begin{cases}(-1)^{\frac{q+1}{2}+n} & \text{if } q \text{ is odd}\\ (-1)^{\frac{q}{2}+n} & \text{if } q \text{ is even}\end{cases}\,. \tag{125}$$

Using [8, Sec. 6.1.15], for $n \geqslant 1$ and $q \geqslant 2n+3$, we get

$$\begin{aligned}\beta_{q\,n} = &\frac{\epsilon_{q\,n}}{2(q-1)(q+1)\Gamma\left(\frac{q}{2}+n+1\right)\Gamma\left(\frac{q}{2}-n+1\right)}\\ &\times\Bigg[(q+1)\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right)\\ &+(q+1)\left(\frac{q}{2}-n\right)\left(\frac{q}{2}-n-1\right)-2(q-1)n^2\Bigg].\end{aligned} \tag{126}$$

It follows from $q \geqslant 2n+3$ that the denominator of (126) is positive, and that

$$(q+1)\left(\frac{q}{2}-n\right)\left(\frac{q}{2}-n-1\right) > 0 \tag{127}$$

and

$$(q+1)\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right) > 2(q-1)n^2\,. \tag{128}$$

This and (126), allow us to write

$$\mathrm{sgn}\left(\beta_{q\,n}\right) = \mathrm{sgn}\left(\epsilon_{q\,n}\right)\,. \tag{129}$$

The results (126) and (129) allow us to assert that, for all $n \geqslant 1$ and all positive $x \in \mathbb{R}$:

- the terms $\beta_{(2p+1)\,n}\ x^p$ of the series $\mathrm{s}_n^{\mathrm{Odd}}(x)$ tend to 0 as $p \to \infty$;
- the terms $\beta_{(2p)\,n}\ x^p$ of the series $\mathrm{s}_n^{\mathrm{Even}}(x)$ tend to 0 as $p \to \infty$; and
- the series $\mathrm{s}_n^{\mathrm{Odd}}(x)$ and $\mathrm{s}_n^{\mathrm{Even}}(x)$ are alternating.

We now study the sign of $|\beta_{q\,n}| - |\beta_{(q+2)\,n}\ x|$ for $x > 0$, $n \geqslant 1$ and $q \geqslant 2n+3$. Using [8, Sec. 6.1.15] and (126), we get

$$\begin{aligned}|\beta_{q\,n}| - |\beta_{(q+2)\,n}\ x| > &\frac{1}{D_{q\,n}}\\ \times\Bigg(&(q+1)\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right)\left(\frac{q}{2}-n+1\right)\\ &-2(q-1)\,n^2\left(\frac{q}{2}-n+1\right)\\ &-x(q-1)\left[\frac{q}{2}+n+\frac{\left(\frac{q}{2}-n+1\right)\left(\frac{q}{2}-n\right)}{\frac{q}{2}+n+1}\right]\Bigg),\end{aligned} \tag{130}$$

where

$$\begin{aligned}&D_{q\,n} =\\ &\quad 2(q-1)(q+1)\Gamma\left(\frac{q}{2}+n+1\right)\Gamma\left(\frac{q}{2}-n+2\right).\end{aligned} \tag{131}$$

For $n \geqslant 1$, we have

$$\frac{q}{2}+n > \frac{\left(\frac{q}{2}-n+1\right)\left(\frac{q}{2}-n\right)}{\frac{q}{2}+n+1}\,. \tag{132}$$

Thus, for $n \geqslant 1$, $q \geqslant 2n+3$ and $x > 0$, we have

$$|\beta_{q\,n}| - |\beta_{(q+2)\,n}\,x| > \frac{1}{D_{q\,n}} \times \Bigg((q+1)\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right)\left(\frac{q}{2}-n+1\right) - 2(q-1)\left\{ n^2\left(\frac{q}{2}-n+1\right) + x\left(\frac{q}{2}+n\right)\right\}\Bigg). \tag{133}$$

We observe that $q \geqslant 2n+3$ allows us to write

$$\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right) \geqslant 4n^2+4n+\frac{3}{4}\,, \tag{134}$$

hence

$$\frac{1}{2}(q+1)\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right)\left(\frac{q}{2}-n+1\right) > 2(q-1)\,n^2\left(\frac{q}{2}-n+1\right) \tag{135}$$

Using (135) in (133), we get

$$|\beta_{q\,n}| - |\beta_{(q+2)\,n}\,x| > \frac{1}{D_{q\,n}} \times \Bigg(\frac{1}{2}(q+1)\left(\frac{q}{2}+n\right)\left(\frac{q}{2}+n-1\right)\left(\frac{q}{2}-n+1\right) - 2(q-1)\,x\left(\frac{q}{2}+n\right)\Bigg). \tag{136}$$

For a positive $x \in \mathbb{R}$, we define the integer

$$P_n^{\min}(x) = \operatorname{ceil}\left(\sqrt{4x+(n-1)^2}\right). \tag{137}$$

For $n \geqslant 1$, $q \geqslant 2n+3$ and $x > 0$, any term of degree $p \geqslant P_n^{\min}(x)$ of $\mathrm{s}_n^{\mathrm{Odd}}(x)$ or $\mathrm{s}_n^{\mathrm{Even}}(x)$ has a coefficient $\beta_{q\,n}$ such that

$$q \geqslant 2\sqrt{4x+(n-1)^2}\,, \tag{138}$$

hence

$$\left(\frac{q}{2}+n-1\right)\left(\frac{q}{2}-n+1\right) \geqslant 4x \tag{139}$$

and

$$|\beta_{q\,n}| - |\beta_{(q+2)\,n}\,x| > 0\,, \tag{140}$$

where we have used (136). Thus, $P_n^{\min}(x)$ is such that: the absolute values of the terms of degree $p \geqslant P_n^{\min}(x)$ of $\mathrm{s}_n^{\mathrm{Odd}}(x)$ form a decreasing sequence; and the absolute values of the terms of degree $p \geqslant P_n^{\min}(x)$ of $\mathrm{s}_n^{\mathrm{Even}}(x)$ form a decreasing sequence.

Thus, if $P'_n(x)$ and $P''_n(x)$ are two integers greater than or equal to $P_n^{\min}(x)$, we can use [13, Sec. 1.3.3] to assert that, if $2P'_n(x) \geqslant 2n+3$ and $2P''_n(x) \geqslant 2n+2$, then

$$\left|\sum_{p=P'_n(x)}^{\infty} \beta_{(2p)\,n}\,x^p\right| \leqslant \left|\beta_{(2P'_n(x))\,n}\,x^{P'_n(x)}\right| \tag{141}$$

and

$$\left|\sum_{p=P''_n(x)}^{\infty} \beta_{(2p+1)\,n}\,x^p\right| \leqslant \left|\beta_{(2P''_n(x)+1)\,n}\,x^{P''_n(x)}\right|. \tag{142}$$

In the right-hand side of (110), the first non-zero coefficient of one of the series is $\beta_{(d_{\mathrm{A}}(n)+1)\,n}$, and the first non-zero coefficient of the other series is $\beta_{(d_{\mathrm{A}}(n)+2)\,n}$. Thus, it follows from (110), (137) and (141)–(142) that, for $n \geqslant 1$ and a specified integer $d_{\mathrm{A}}(n)$ such that

$$d_{\mathrm{A}}(n) \geqslant \max\left(2\operatorname{ceil}\left(\sqrt{4(ka)^2+(n-1)^2}\right), 2n+2\right), \tag{143}$$

we have

$$\left|\sum_{q=d_{\mathrm{A}}(n)+1}^{\infty} \alpha_{q\,n}(ka)^q\right| \leqslant \mathfrak{E}_n\left(ka, d_{\mathrm{A}}(n)\right), \tag{144}$$

where, for any integer $q \geqslant 1$,

$$\mathfrak{E}_n(ka,q) = \sqrt{\begin{aligned}&\left[\left|\alpha_{(q+1)\,n}\right|(ka)^{q+1}\right]^2\\ &+\left[\left|\alpha_{(q+2)\,n}\right|(ka)^{q+2}\right]^2\end{aligned}}\,. \tag{145}$$

We note that, if $n \geqslant (ka)^2$, then

$$(n+1)^2-(n-1)^2 = 4n \geqslant 4(ka)^2\,, \tag{146}$$

hence

$$n+1 \geqslant \operatorname{ceil}\left(\sqrt{4(ka)^2+(n-1)^2}\right). \tag{147}$$

Consequently,

$$\left[\left(d_{\mathrm{A}}(n) \geqslant 2n+2\right) \text{ and } \left(n \geqslant (ka)^2\right)\right] \Longrightarrow \left[d_{\mathrm{A}}(n) \geqslant 2\operatorname{ceil}\left(\sqrt{4(ka)^2+(n-1)^2}\right)\right]. \tag{148}$$

Thus, if $d_{\mathrm{A}}(n) \geqslant 2(n+1)$ and $n \geqslant (ka)^2$, then the condition (143) is satisfied.

The main results of this Appendix C are (121)–(123), (143)–(145) and (148).

APPENDIX D

In this Appendix D, p being a positive integer, we study the sequence $p \mapsto \Xi_p$ defined by (10).

Let n be a positive integer. The n-th harmonic number is

$$H_n = \sum_{k=1}^{n} \frac{1}{k}\,. \tag{149}$$

It is asymptotically given by [15, Sec. 10.4]:

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^6}\right), \tag{150}$$

where we have used Landau's Big-O notation.

We observe that

$$\sum_{m=1}^{2p} \frac{1}{m} = \sum_{m=0}^{p-1} \frac{1}{2m+1} + \frac{1}{2}\sum_{m=1}^{p} \frac{1}{m}\,. \tag{151}$$

Using (150) twice in (151), we obtain

$$\ln(2p) + \gamma + \frac{1}{4p} - \frac{1}{48p^2} + O\left(\frac{1}{p^4}\right) = \sum_{m=0}^{p-1} \frac{1}{2m+1} + \frac{1}{2}\left[\ln p + \gamma + \frac{1}{2p} - \frac{1}{12p^2} + O\left(\frac{1}{p^4}\right)\right], \tag{152}$$

TABLE 4. Some coefficients Ξ_p and related quantities.

p	$\xi_{\min}(p)$	Ξ_p	$\xi_{\max}(p)$	$-1/24p^2$	r_p
1	-0.05208	-0.03649	-0.00839	-0.04167	14.2%
2	-0.01308	-0.01001	-0.00534	-0.01042	4.07%
3	-0.00566	-0.00454	-0.00301	-0.00463	1.88%
4	-0.00310	-0.00258	-0.00189	-0.00260	1.07%
5	-0.00194	-0.00166	-0.00129	-0.00167	0.69%
6	-0.00133	-0.00115	-0.00094	-0.00116	0.48%
7	-0.00096	-0.00085	-0.00071	-0.00085	0.35%
8	-0.00073	-0.00065	-0.00056	-0.00065	0.27%
9	-0.00057	-0.00051	-0.00045	-0.00051	0.21%
10	-0.00046	-0.00042	-0.00037	-0.00042	0.17%

and then

$$\ln(4p) + \gamma + \frac{1}{24p^2} + O\left(\frac{1}{p^4}\right) = 2\sum_{m=0}^{p-1} \frac{1}{2m+1}\,. \tag{153}$$

It follows that an asymptotic approximation of Ξ_p is

$$\begin{aligned}\Xi_p &= \ln(4p) + \gamma - 2\sum_{m=0}^{p-1} \frac{1}{2m+1} \\ &= -\frac{1}{24p^2} + O\left(\frac{1}{p^4}\right).\end{aligned} \tag{154}$$

We want now to compute a lower bound and an upper bound on Ξ_p. Using [15, Sec. 9.3], we get

$$\frac{1}{24(p+1)^2} < H_p - \ln\left(p+\frac{1}{2}\right) - \gamma < \frac{1}{24p^2} \tag{155}$$

and

$$\frac{1}{24(2p+1)^2} < H_{2p} - \ln\left(2p+\frac{1}{2}\right) - \gamma < \frac{1}{96p^2}. \tag{156}$$

By (151) we have

$$-2\sum_{m=0}^{p-1} \frac{1}{2m+1} = H_p - 2H_{2p}\,, \tag{157}$$

hence

$$\begin{aligned}&-\gamma - \ln\frac{(2p+0.5)^2}{p+0.5} + \frac{1}{24(p+1)^2} - \frac{1}{48p^2} \\ &\quad< -2\sum_{m=0}^{p-1} \frac{1}{2m+1} \\ &< -\gamma - \ln\frac{(2p+0.5)^2}{p+0.5} + \frac{1}{24p^2} - \frac{1}{48(p+0.5)^2}\,.\end{aligned} \tag{158}$$

Using

$$\ln\frac{(2p+0.5)^2}{p+0.5} = \ln(4p) + \ln\left(1+\frac{1}{16p(p+0.5)}\right) \tag{159}$$

and the fact that the series expansion of $\ln(1+x)$ for $|x|<1$ is an alternating series, we get

$$\begin{aligned}&\ln(4p) + \frac{1}{16p(p+0.5)} - \frac{1}{512p^2(p+0.5)^2} \\ &\quad< \ln\frac{(2p+0.5)^2}{p+0.5} < \ln(4p) + \frac{1}{16p(p+0.5)}\,.\end{aligned} \tag{160}$$

Using (160) and (158) in (10), we obtain

$$\xi_{\min}(p) < \Xi_p < \xi_{\max}(p)\,, \tag{161}$$

where the bounds are

$$\xi_{\min}(p) = -\frac{1}{16p(p+0.5)} + \frac{1}{24(p+1)^2} - \frac{1}{48p^2} \tag{162}$$

and

$$\begin{aligned}\xi_{\max}(p) = &-\frac{1}{16p(p+0.5)} \\ &+ \frac{1}{512p^2(p+0.5)^2} + \frac{1}{24p^2} - \frac{1}{48(p+0.5)^2}\,.\end{aligned} \tag{163}$$

Table 4 shows $\xi_{\min}(p)$, Ξ_p, $\xi_{\max}(p)$, the approximation

$$\Xi_p \simeq -\frac{1}{24p^2} \tag{164}$$

and the absolute value r_p of the relative error of this approximation. Let f be the function defined on $\mathbb{R}_+^*$ by $\forall x \in \mathbb{R}_+^*$,

$$f(x) = \ln\left(1+\frac{1}{x}\right) - \frac{2}{2x+1}\,, \tag{165}$$

which, by (10), is such that

$$\Xi_{p+1} - \Xi_p = f(p)\,. \tag{166}$$

For all $x \in \mathbb{R}_+^*$, the derivative of f at x is given by

$$f'(x) = \frac{-1}{x(x+1)(2x+1)^2}\,. \tag{167}$$

Since $f'(x) < 0$ for all $x \in \mathbb{R}_+^*$, and $f(x) \to 0$ as $x \to \infty$, we have $f(x) > 0$ for all $x \in \mathbb{R}_+^*$. Accordingly, the sequence $q \mapsto \Xi_q$ defined for $q \geqslant 1$ is strictly increasing. Since it tends to 0 as n becomes large, we have $\Xi_p < 0$ for any positive integer p. Thus, using (162), for any positive $p \in \mathbb{N}$, we get:

$$\xi_{\min}(p) = \frac{-2p^3 - 7.5p^2 - 5p - 0.5}{48p^2(p+0.5)(p+1)^2} < \Xi_p < 0\,. \tag{168}$$

If $p \geqslant 1$, then $2p^3 + 7.5p^2 + 5p + 0.5 < 2(p+1)^2(p+2)$, hence

$$\xi_{\min}(p) > -\frac{p+2}{24p^2(p+0.5)}\,, \tag{169}$$

$$\xi_{\min}(p) > -\frac{p+2}{24p^3} \tag{170}$$

and

$$0 < -\Xi_p < \frac{p+2}{24p^3} \tag{171}$$

for any positive integer p.

Copyright ©2026 by Excem

APPENDIX E

Let $t \in \mathbb{R}_+^*$. In this Appendix E, we want to show that, q being a positive integer, the sequence $q \mapsto r_q(t)$ such that

$$r_q(t) = \frac{|\Xi_q|}{\mathrm{K}_0\,(qt)\,\mathrm{I}_0\,(qt)} \tag{172}$$

is less than a positive and strictly decreasing sequence that approaches 0 as $q \to \infty$.

By (171), we have

$$0 < |\Xi_q| < \frac{q+2}{24q^3}\,. \tag{173}$$

According to Corollary 2 of [16], for any $x \in \mathbb{R}_+^*$, we have

$$\mathrm{K}_0(x)\mathrm{I}_0(x) > \frac{1}{2\sqrt{x^2+\dfrac{1}{3}}}\,. \tag{174}$$

We consider the function $\tilde{r}$ defined for any $\nu \in \mathbb{R}_+^*$ by

$$\tilde{r}(\nu,t) = \frac{\nu+2}{12\,\nu^3}\sqrt{(\nu t)^2+\frac{1}{3}} \tag{175}$$

It follows from (172)–(175) that, for any positive integer q, we have

$$0 < r_q(t) < \tilde{r}(q,t)\,. \tag{176}$$

We get

$$\begin{aligned}\frac{\partial\,\tilde{r}(\nu,t)}{\partial\nu} &= \frac{1}{12}\sqrt{(\nu t)^2+\frac{1}{3}}\\ &\times\left(\frac{\nu t^2}{(\nu t)^2+\dfrac{1}{3}}\,\frac{\nu+2}{\nu^3}+\frac{-2\nu-6}{\nu^4}\right)\end{aligned} \tag{177}$$

and then

$$\frac{\partial\,\tilde{r}(\nu,t)}{\partial\nu} = \frac{(\nu t)^2(\nu+2)-2(\nu+3)\left[(\nu t)^2+\dfrac{1}{3}\right]}{12\,\nu^4\sqrt{(\nu t)^2+\dfrac{1}{3}}}\,. \tag{178}$$

It follows that

$$\frac{\partial\,\tilde{r}(\nu,t)}{\partial\nu} = \frac{-(\nu t)^2(\nu+4)-\dfrac{2}{3}(\nu+3)}{12\,\nu^4\sqrt{(\nu t)^2+\dfrac{1}{3}}}\,. \tag{179}$$

Hence,

$$\frac{\partial\,\tilde{r}(\nu,t)}{\partial\nu} < 0. \tag{180}$$

Thus, the function $\nu \mapsto \tilde{r}(\nu,t)$ is strictly decreasing. Moreover, this positive function approaches 0 as ν becomes arbitrarily large. Accordingly, it follows from (176) that the sequence $q \mapsto r_q(t)$ is less than the positive and strictly decreasing sequence $q \mapsto \tilde{r}(q,t)$ that approaches 0 as q becomes arbitrarily large.

APPENDIX F

In this Appendix F, we want to show that the function Q defined by $Q(x) = x^2\,\mathrm{K}_0(x)\,\mathrm{I}_0(x)$ for all $x \in \mathbb{R}_+^*$ is strictly increasing. We also want to compute an easily integrable lower bound on the derivative of Q at x.

Let $x \in \mathbb{R}_+^*$. Using [8, Eq. 9.6.27], we find that the derivative of Q is

$$Q'(x) = x\mathrm{K}_0(x)\,\mathrm{I}_0(x)\left[2+x\frac{\mathrm{I}_1(x)}{\mathrm{I}_0(x)}-x\frac{\mathrm{K}_1(x)}{\mathrm{K}_0(x)}\right]. \tag{181}$$

The Wronskian of I_0 and K_0 is known [8, Eq. 9.6.15], and given by $\mathrm{K}_1(x)\,\mathrm{I}_0(x)+\mathrm{K}_0(x)\,\mathrm{I}_1(x) = 1/x$. Using this result in (181), we get

$$Q'(x) = x\,\mathrm{K}_0(x)\,\mathrm{I}_0(x)\,f(x), \tag{182}$$

where

$$f(x) = 2+2x\frac{\mathrm{I}_1(x)}{\mathrm{I}_0(x)}-\frac{1}{\mathrm{K}_0(x)\mathrm{I}_0(x)}\,. \tag{183}$$

According to Theorem 2 of [17], we have

$$\frac{\mathrm{I}_1(x)}{\mathrm{I}_0(x)} > \frac{x}{1+\sqrt{1+x^2}}\,. \tag{184}$$

Using (174) and (184) in (183), we obtain

$$f(x) > 2+\frac{2x^2}{1+\sqrt{1+x^2}}-2\sqrt{x^2+\frac{1}{3}}\,, \tag{185}$$

that is to say

$$f(x) > 2\left(\sqrt{x^2+1}-\sqrt{x^2+\frac{1}{3}}\right). \tag{186}$$

By (186), $f(x)$ is positive. Since $x\,\mathrm{K}_0(x)\mathrm{I}_0(x)$ is also positive by [8, Sec. 9.6.1], it follows from (182) that $Q'(x)$ is positive. Accordingly, Q is strictly increasing in $\mathbb{R}_+^*$.

To obtain a convenient lower bound on $Q'(x)$, we combine (174) and (182) and (186) to obtain

$$Q'(x) > \frac{2x}{3\sqrt{x^2+1}\sqrt{x^2+\dfrac{1}{3}}+3x^2+1}\,. \tag{187}$$

Hence, for any $x \in \mathbb{R}_+^*$, we obtain

$$Q'(x) > \frac{x}{3x^2+2}\,, \tag{188}$$

which qualifies as an easily integrable lower bound on $Q'(x)$.

APPENDIX G

In this Appendix G, n being a positive integer, we study the sequence $n \mapsto D_n(t) = Q(nt)/t^2 + n^2\,\Xi_n$, in which Ξ_n is defined by (10), Q is the function defined in Section IV.C and Appendix F, and $t \in \mathbb{R}_+^*$ satisfies $t \leqslant 1/3$. We have

$$\begin{aligned}&(n+1)^2\,\Xi_{n+1}-n^2\,\Xi_n =\\ &\quad(n+1)^2\left(\Xi_{n+1}-\Xi_n\right)+\left((n+1)^2-n^2\right)\Xi_n =\\ &\qquad(n+1)^2\left(\Xi_{n+1}-\Xi_n\right)+(2n+1)\,\Xi_n\,.\end{aligned} \tag{189}$$

Copyright ©2026 by Excem

Using (165)–(166) and (168)–(169) in (189), we get

$$(n+1)^2\,\Xi_{n+1}-n^2\,\Xi_n > (n+1)^2\left(\ln\left(1+\frac{1}{n}\right)-\frac{2}{2n+1}\right) - (2n+1)\,\frac{n+2}{24n^2(n+0.5)}\,. \tag{190}$$

In the case $n=1$, Table 4 allows us to obtain a better lower bound than (190):

$$(n+1)^2\,\Xi_{n+1}-n^2\,\Xi_n = 4\,\Xi_2-\Xi_1 > -0.005\,. \tag{191}$$

In the case $n \geqslant 2$, since the power series expansion of $\ln(1+x)$ is an alternating series if $0 \leqslant x < 1$, (190) entails

$$(n+1)^2\,\Xi_{n+1}-n^2\,\Xi_n > (n+1)^2\left(\frac{1}{n}-\frac{1}{2n^2}-\frac{2}{2n+1}\right)-\frac{n+2}{12n^2}\,, \tag{192}$$

$$(n+1)^2\,\Xi_{n+1}-n^2\,\Xi_n > \frac{-6(n+1)^2-(n+2)(2n+1)}{12n^2\,(2n+1)} \tag{193}$$

and

$$(n+1)^2\,\Xi_{n+1}-n^2\,\Xi_n > -\frac{8n^2+17n+8}{12n^2\,(2n+1)}\,. \tag{194}$$

We have

$$Q((n+1)t)-Q(nt)=\int_{nt}^{(n+1)t} Q'(x)\,dx \tag{195}$$

Using (188) in (195), we get

$$Q((n+1)t)-Q(nt) > \frac{1}{6}\ln\frac{3(n+1)^2t^2+2}{3n^2t^2+2} \tag{196}$$

and then

$$Q((n+1)t)-Q(nt) > \frac{1}{6}\ln\left(1+\frac{3(2n+1)t^2}{3n^2t^2+2}\right)\,. \tag{197}$$

Since $t \leqslant 1/3$, we have $9t^2 \leqslant 1$, hence, if $n=1$, then $3(2n+1)t^2 < 3n^2t^2+2$. If $n \geqslant 2$, we have $2nt^2 \leqslant n^2t^2$, hence $t \leqslant 1/3$ leads us to $(2n+1)t^2 \leqslant n^2t^2+1/9$, and then to $3(2n+1)t^2 < 3n^2t^2+2$. Thus, for any positive integer n, we have $3(2n+1)t^2 < 3n^2t^2+2$. Since the series expansion of $\ln(1+x)$ for $|x|<1$ is an alternating series, we obtain

$$Q((n+1)t)-Q(nt) > \frac{1}{2}\left(\frac{(2n+1)t^2}{3n^2t^2+2}-\frac{3}{2}\,\frac{((2n+1)t^2)^2}{(3n^2t^2+2)^2}\right)\,, \tag{198}$$

and then

$$\frac{1}{t^2}\big[Q((n+1)t)-Q(nt)\big] > \frac{6t^2n^3-3t^2n^2+(4-6t^2)n+2-1.5\,t^2}{2(9n^4t^4+12n^2t^2+4)}\,. \tag{199}$$

We now want to determine the sign of $D_{n+1}(t)-D_n(t)$. In the case $n=1$, using (191) and (199) in the definition of $D_n(t)$, we obtain

$$D_2(t)-D_1(t) > \frac{-4.5t^2+6}{18t^4+24t^2+8}-0.005 \tag{200}$$

and

$$D_2(t)-D_1(t) > \frac{-0.09t^4-4.62t^2+5.96}{18t^4+24t^2+8} \tag{201}$$

Since we have assumed $0 < t \leqslant 1/3$, it follows from (201) that

$$D_2(t)-D_1(t) > 0 \tag{202}$$

In the case $n \geqslant 2$, using (194) and (199) in the definition of $D_n(t)$, we obtain

$$D_{n+1}(t)-D_n(t) > \frac{\mathcal{P}_n(t)}{\mathcal{R}_n(t)}\,, \tag{203}$$

where

$$\begin{aligned}\mathcal{P}_n(t) ={}& \\ &\big(6t^2n^3-3t^2n^2+(4-6t^2)n+2-1.5\,t^2\big)\big(24n^3+12n^2\big)\\ &-\big(8n^2+17n+8\big)\big(18t^4n^4+24t^2n^2+8\big)\end{aligned} \tag{204}$$

and $\mathcal{R}_n(t)=24n^2(2n+1)(9n^4t^4+12n^2t^2+4)>0$.

We find:

$$\begin{aligned}\mathcal{P}_n(t) ={}& \\ &144t^2(1-t^2)n^6-306t^4n^5+(96-372t^2-144t^4)n^4\\ &+(96-516t^2)n^3-(40+210t^2)n^2-136n-64.\end{aligned} \tag{205}$$

Using $t^2 < 1/9$ and $n \geqslant 2$, we obtain: $144(1-t^2) > 128$; then $144t^2(1-t^2)n^6 > 256t^2n^5$; then $306t^4n^5 < 34t^2n^5$; then $144t^2(1-t^2)n^6-306t^4n^5 > 444t^2n^4$; then $144t^2 < 16$; then $96-372t^2-144t^4 > 96-388t^2$; then

$$\begin{aligned}&144t^2(1-t^2)n^6-306t^4n^5+(96-372t^2-144t^4)n^4 > \\ &\qquad (96+56t^2)n^4 > (192+112t^2)n^3\,;\end{aligned} \tag{206}$$

$$\begin{aligned}&144t^2(1-t^2)n^6-306t^4n^5+(96-372t^2-144t^4)n^4\\ &\qquad +(96-516t^2)n^3 > (576-808t^2)n^2\,;\end{aligned} \tag{207}$$

$$\begin{aligned}&144t^2(1-t^2)n^6-306t^4n^5+(96-372t^2-144t^4)n^4\\ &\qquad +(96-516t^2)n^3-(40+210t^2)n^2 > \\ &\qquad\qquad (1072-2036t^2)n\end{aligned} \tag{208}$$

and

$$\mathcal{P}_n(t) > 1808-4072t^2 > 1356\,. \tag{209}$$

It follows from (202), (203) and (209) that, for any positive integer n, we have

$$D_{n+1}(t)-D_n(t) > 0\,. \tag{210}$$

APPENDIX H

In this Appendix H, n is an integer such that $n \geqslant 2$ and we want to compare the absolute value of the power series $\mathrm{S}_n(ka)$ defined by (13) to $1/n$ as n becomes large, under the assumption that the frequency f satisfies $f \leqslant f_{\max}$, or equivalently that $ka \leqslant 2.5$.

To this end, we write that $\mathrm{s}_n^{\mathrm{poly}}(ka)$ defined by (101) is given by

$$\mathrm{s}_n^{\mathrm{poly}}(ka)=\mathrm{s}_n^{(1)}(ka)+\mathrm{s}_n^{(2)}(ka)\,, \tag{211}$$

where

$$\mathrm{s}_n^{(1)}(ka) = \sum_{q=2}^{2n-2} \alpha_{q\,n}(ka)^q \,, \tag{212}$$

and

$$\mathrm{s}_n^{(2)}(ka) = \sum_{q=2n-1}^{2n+2} \alpha_{q\,n}(ka)^q \,, \tag{213}$$

To begin with, we seek an upper bound of $\mathrm{s}_n^{(1)}(ka)$. For any positive integers n and q such that $2 \leqslant q \leqslant 2n-2$, we find that:

- if q is even, then $q-1 \leqslant 2(n-1)-1$ and $q+1 \leqslant 2n-1$, which entails that (88) can be used to obtain $\chi_{(q-1)\,(n-1)} = 0$, $\chi_{(q-1)\,(n+1)} = 0$ and $\chi_{(q+1)\,n} = 0$, hence (12) leads us to
$$\alpha_{q\,n} = 0\,; \tag{214}$$
- if q is odd, then we have $q-1 \leqslant 2(n-1)-2$ and $q+1 \leqslant 2n-2$, which entails that (93) can be used to compute $\chi_{(q-1)\,(n-1)}$, $\chi_{(q-1)\,(n+1)}$ and $\chi_{(q+1)\,n}$, hence (12) leads us to

$$\begin{aligned}\alpha_{q\,n} = {} & \frac{2^{q-1}}{\pi \displaystyle\prod_{i=n+0.5-q/2}^{n-1.5+q/2} (2i+1)} \\ & \times \left[\frac{2n+q-2}{2(q-1)(2n-q)} + \frac{2n-q+2}{2(q-1)(2n+q)} \right. \\ & \left. - \frac{4n^2}{(q+1)(2n-q)(2n+q)} \right] .\end{aligned} \tag{215}$$

It follows from (215) that, for any positive integer n and any odd integer q such that $2 \leqslant q \leqslant 2n-2$, we have

$$\begin{aligned}\alpha_{q\,n} = {} & \frac{2^{q-1}}{\pi \displaystyle\prod_{i=n+0.5-q/2}^{n-1.5+q/2} (2i+1)} \\ & \times \frac{8n^2 + q(q+1)(q-2)}{(q-1)(q+1)(2n-q)(2n+q)} \,,\end{aligned} \tag{216}$$

which is positive because $q-2 \geqslant 1$ and $2n-q \geqslant 3$. Let us consider the function P_n of the real variables x used to replace the discrete variable q, such that

$$\mathrm{P}_n(x) = (x^2-1)(4n^2-x^2)\,. \tag{217}$$

Since $\mathrm{P}_n(x) = -X^2 + X(4n^2+1) - 4n^2$ where $X = x^2$, this function has a maximum at x equal to x_{M} given by

$$x_{\mathrm{M}} = \sqrt{\frac{4n^2+1}{2}}\,. \tag{218}$$

We are looking for a lower bound of $\mathrm{P}_n(x)$ for x lying in the interval $I_n = [3, 2n-3]$, and $n \geqslant 3$. We note that $n \geqslant 3$ entails $x_{\mathrm{M}} \geqslant 3$. Thus, x_{M} lies in I_n if and only if $2n^2 + 1/2 \leqslant 4n^2 - 12n + 9$. The polynomial $2y^2 - 12y + 17/2$, in the real variable y, having two roots y_1 and y_2 such that $y_1 \simeq 0.82$ and $y_2 \simeq 5.18$, x_{M} lies in I_n if and only if $n \geqslant 6$. Accordingly, if $3 \leqslant n \leqslant 5$, then P_n is an increasing function over I_n, hence $\forall x \in I_n, \mathrm{P}_n(x) \geqslant \mathrm{P}_n(3)$. In contrast, if $n \geqslant 6$, we need to compare $\mathrm{P}_n(3)$ to $\mathrm{P}_n(2n-3)$, because the smallest is a lower bound of $\mathrm{P}_n(x)$ for $x \in I_n$. We have $\mathrm{P}_n(2n-3) - \mathrm{P}_n(3) = 4n(12n^2 - 53n + 51)$. The polynomial $y(12y^2 - 53y + 51)$ in the real variable y having three roots y_1, y_2 and y_3 such that $y_1 = 0$, $y_2 \simeq 1.42$ and $y_3 = 3$, it follows that, if $n \geqslant 6$, then $\mathrm{P}_n(2n-3) > \mathrm{P}_n(3)$, hence $\forall x \in I_n, \mathrm{P}_n(x) \geqslant \mathrm{P}_n(3)$.

Thus, if $n \geqslant 3$, then $\forall x \in I_n$ we have

$$\mathrm{P}_n(x) \geqslant \mathrm{P}_n(3) = 8(4n^2-9) \geqslant 24n^2\,. \tag{219}$$

Consequently, for any integer n such that $n \geqslant 3$ and any odd integer q such that $2 \leqslant q \leqslant 2n-2$, we have

$$0 < \frac{8n^2}{(q-1)(q+1)(2n-q)(2n+q)} \leqslant \frac{1}{3} \tag{220}$$

For any positive integer n and any odd integer q such that $2 \leqslant q \leqslant 2n-2$, we have

$$\frac{q(q+1)(q-2)}{(q-1)(q+1)(2n-q)(2n+q)} \leqslant \frac{q}{4n^2-q^2}\,. \tag{221}$$

Let us consider the function Q_n of the real variables x used to replace the discrete variable q, such that

$$\mathrm{Q}_n(x) = \frac{x}{4n^2-x^2}\,, \tag{222}$$

defined for any $x \in \mathbb{R}$ such that $x \neq 2n$ and $x \neq -2n$. Where Q_n is defined, its derivative is positive hence Q_n is increasing. In the interval $I_n = [3, 2n-3]$, Q_n is defined and increasing, hence

$$\mathrm{Q}_n(x) \leqslant \mathrm{Q}_n(2n-3) = \frac{2n-3}{12n-9} < \frac{1}{6}\,. \tag{223}$$

Using (221)–(223), we find that, for any positive integer n and any odd integer q such that $2 \leqslant q \leqslant 2n-2$, we have

$$0 < \frac{q(q+1)(q-2)}{(q-1)(q+1)(2n-q)(2n+q)} < \frac{1}{6}\,. \tag{224}$$

Using (220) and (224) in (216), we get: for any integer $n \geqslant 3$ and any odd integer q such that $2 \leqslant q \leqslant 2n-2$,

$$0 < \alpha_{q\,n} < \frac{2^{q-2}}{\pi \displaystyle\prod_{i=n+0.5-q/2}^{n-1.5+q/2} (2i+1)}\,. \tag{225}$$

We now observe that, for any integer $n \geqslant 3$ and any odd integer q such that $3 \leqslant q \leqslant 2n-3$,

$$p_{q\,n} = \prod_{i=n+0.5-q/2}^{n-1.5+q/2} (2i+1) \tag{226}$$

is a product of $q-1$ factors, which include the factors $2n-1$ and $2n+1$, each of the $q-1$ factors being larger than or equal to 5. Hence

$$\prod_{i=n+0.5-q/2}^{n-1.5+q/2} (2i+1) \geqslant (2n-1)\,(2n+1)\,5^{q-3}\,. \tag{227}$$

Copyright © 2026 by Excem

If the condition $f \leqslant f_{\max}$ is satisfied, $2ka \leqslant 5$, hence, for any $n \geqslant 3$ and any odd integer q such that $2 \leqslant q \leqslant 2n-2$, (225) leads us to

$$0 < \alpha_{q\,n}(ka)^q < \frac{2(ka)^3\, 5^{q-3}}{\pi \displaystyle\prod_{i=n+0.5-q/2}^{n-1.5+q/2} (2i+1)}\,, \tag{228}$$

and then (227) to

$$0 < \alpha_{q\,n}(ka)^q < \frac{2(ka)^3}{\pi\,(2n-1)\,(2n+1)}\,. \tag{229}$$

For $n \geqslant 3$, the sum in (212) has $2n-3$ terms, among which $n-1$ are zero by (214), the $n-2$ other terms corresponding to odd integers q such that $2 \leqslant q \leqslant 2n-2$. Thus, if the condition $f \leqslant f_{\max}$ is satisfied, for any integer $n \geqslant 3$, said $n-2$ other terms satisfy (229), hence

$$0 \leqslant \mathrm{s}_n^{(1)}(ka) < \frac{(ka)^3}{\pi\,(2n+1)}\,. \tag{230}$$

We now seek an upper bound of $\mathrm{s}_n^{(2)}(ka)$, which is a polynomial in the variable ka, of coefficients $\alpha_{(2n-1)\,n}$, $\alpha_{(2n)\,n}$, $\alpha_{(2n+1)\,n}$ and $\alpha_{(2n+2)\,n}$. According to [8, Ch. 6], the function $x \mapsto \Gamma(x)$ of the real variable x is increasing for $x > 1.465$, and it satisfies $\Gamma(0.5) > 1.7$ and $\Gamma(-1.5) > 2.3$. Thus, since we assumed $n \geqslant 2$, it follows from (8) that

$$\left|\chi_{(2n-2)\,(n-1)}\right| < \frac{1}{(2n-2)!}\,, \tag{231}$$

$$\left|\chi_{(2n-2)\,(n+1)}\right| < \frac{1}{2(2n-2)\,(2n-1)!} \tag{232}$$

and

$$\left|\chi_{(2n)\,n}\right| < \frac{1}{(2n)!}\,, \tag{233}$$

hence (12) leads us to

$$\left|\alpha_{(2n-1)\,n}\right| < \frac{1+\dfrac{1+2n(2n-2)}{2(2n-1)(2n-2)}}{2(2n-2)!} \tag{234}$$

and

$$\left|\alpha_{(2n-1)\,n}\right| < \frac{1}{(2n-2)!}\,. \tag{235}$$

It follows from (9) that

$$\left|\chi_{(2n-1)\,(n-1)}\right| = \frac{1}{(2n-1)!}\,, \tag{236}$$

$$\chi_{(2n-1)\,(n+1)} = 0 \tag{237}$$

and

$$\left|\chi_{(2n+1)\,n}\right| = \frac{1}{(2n+1)!}\,, \tag{238}$$

hence (12) leads us to

$$\left|\alpha_{(2n)\,n}\right| < \frac{1}{(2n-1)!}\,. \tag{239}$$

Since $\Gamma(1.5) > 0.88$, $\Gamma(-1.5) > 2.3$ and $n \geqslant 2$, it follows from (8) that

$$\left|\chi_{(2n)\,(n-1)}\right| < \frac{1.2}{(2n-1)!}\,, \tag{240}$$

$$\left|\chi_{(2n)\,(n+1)}\right| < \frac{1}{3n\,(2n)!} \tag{241}$$

and

$$\left|\chi_{(2n+2)\,n}\right| < \frac{1.2}{(2n+2)\,(2n)!}\,, \tag{242}$$

hence (12) leads us to

$$\left|\alpha_{(2n+1)\,n}\right| < \frac{1}{(2n-1)!}\,. \tag{243}$$

It follows from (9) that

$$\left|\chi_{(2n+1)\,(n-1)}\right| = \frac{1}{(2n+1)\,(2n-1)!}\,, \tag{244}$$

$$\left|\chi_{(2n+1)\,(n+1)}\right| = 0 \tag{245}$$

and

$$\left|\chi_{(2n+3)\,n}\right| = \frac{1}{(2n+3)\,(2n+1)!}\,, \tag{246}$$

hence (12) leads us to

$$\left|\alpha_{(2n+2)\,n}\right| < \frac{1}{(2n)!}\,. \tag{247}$$

It follows from (213), (235), (239), (243) and (247) that

$$\left|\mathrm{s}_n^{(2)}(ka)\right| < \frac{(ka)^{2n-1}}{(2n-2)!} \times \left(1+\frac{ka}{2n-1}+\frac{(ka)^2}{2n-1}+\frac{(ka)^3}{2n(2n-1)}\right). \tag{248}$$

Thus, if the condition $f \leqslant f_{\max}$ is satisfied, for any integer n such that $n \geqslant 3$, we have

$$\left|\mathrm{s}_n^{(2)}(ka)\right| < 3.3\,\frac{(ka)^{2n-1}}{(2n-2)!}\,, \tag{249}$$

and, since $2 \times 3 \times 4 > (ka)^3$ entails $(2n-3)! > (ka)^{2n-4}$, we also have

$$\left|\mathrm{s}_n^{(2)}(ka)\right| < \frac{3.3\,(ka)^3}{2n-2} \tag{250}$$

for $n \geqslant 4$. We now assume $n \geqslant 4$ and $n \geqslant (ka)^2$, and we posit $d_{\mathrm{A}}(n) = 2(n+1)$. Hence, by (148), the condition (18) is satisfied; by (14), (212) and (213) we have

$$\mathcal{S}_n(ka) = \mathrm{s}_n^{(1)}(ka) + \mathrm{s}_n^{(2)}(ka)\,; \tag{251}$$

and, by (15), (16) and (19) we have

$$\left|\mathrm{S}_n(ka)\right| \leqslant \left|\mathcal{S}_n(ka)\right| + \left|\alpha_{(2n+3)\,n}\right|(ka)^{2n+3} + \left|\alpha_{(2n+4)\,n}\right|(ka)^{2n+4}\,. \tag{252}$$

By $n \geqslant 4$ and (124) we obtain

$$\lim_{n\to\infty} n\left|\alpha_{(2n+3)\,n}\right|(ka)^{2n+3} = 0 \tag{253}$$

and

$$\lim_{n\to\infty} n\left|\alpha_{(2n+4)\,n}\right|(ka)^{2n+4} = 0\,. \tag{254}$$

It follows from (230), (249) and (251)–(254) that, if the condition $f \leqslant f_{\max}$ is satisfied, then

$$\mathrm{S}_n(ka) = O\!\left(\frac{1}{n}\right) \tag{255}$$

as n becomes large. This is the wanted comparison between the absolute value of $\mathrm{S}_n(ka)$ with $1/n$.

REFERENCES

[1] F. Broydé and E. Clavelier, “Some Mathematical Models of a Circular Wire Loop Antenna,” *Excem Research Papers in Electronics and Electromagnetics*, no. 9, doi: 10.5281/zenodo.11407457, Jun. 2024.

[2] F. Broydé and E. Clavelier, “Some Computations on a Single-Turn Circular Loop Antenna”, *Excem Research Notes in Electronics and Electromagnetics*, no. 1, doi: 10.5281/zenodo.12750917, Jul. 2024.

[3] T.T. Wu, “Theory of the thin circular loop antenna,” *J. Mathematical Physics*, J. Mathematical Physics, vol. 3, no. 6, pp. 1301-1304, 1962.

[4] R.W.P. King, C.W. Harrison, Jr. and D.G. Tingley, “The current in bare circular loop antennas in a dissipative medium,” *IEEE Trans. Antennas Propagat.*, vol. AP-13, no. 4, pp. 529-531, Jul. 1965.

[5] R.W.P. King, C.W. Harrison, Jr. and D.G. Tingley, “The admittance of bare circular loop antennas in a dissipative medium,” *IEEE Trans. Antennas Propagat.*, vol. AP-12, no. 4, pp. 434-438, Jul. 1964.

[6] R.W.P. King, *Tables of Antenna Characteristics*, New York, NY, USA: IFI Plenum Data Corporation, 1971.

[7] R.W.P. King, “The loop antenna for transmission and reception,” ch. 11 of *Antennas Theory, Part 1*, R.E. Collin and F.J. Zucker, Ed., New York, NY, USA: McGraw-Hill, 1969.

[8] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, New York, NY, USA: Dover, 1965.

[9] Á. Baricz, “On a product of modified Bessel functions”, *Proc. American Math. Soc.*, vol. 137, no. 1, pp. 189-193, Jan. 2009.

[10] E. Ramis, C. Deschamps and J. Odoux, *Cours de mathématiques spéciales — 3 — Topologie et éléments d’analyse*, 3rd ed., Paris, France: Masson, 1991.

[11] F. Oberhettinger *Fourier Expansions – A Collection of Formulas*, New York, NY, USA: Academic Press, 1973.

[12] E. Hewitt and R.E. Hewitt, “The Gibbs-Wilbraham Phenomenon: An Episode in Fourier Analysis,” *Archive for History of Exact Sciences*, vol. 21, no. 2, pp. 129-160, 1979.

[13] E. Ramis, C. Deschamps and J. Odoux, *Cours de mathématiques spéciales — 4 — Séries et équations différentielles*, Paris, France: Masson, 1977.

[14] F. Broydé and E. Clavelier, “The Open-Circuit Voltage of a Planar Wire Loop Antenna Used for Reception,” *Excem Research Papers in Electronics and Electromagnetics*, no. 6, doi: 10.5281/zenodo.7498910, Jan. 2023.

[15] J. Havil, *Gamma: Exploring Euler’s Constant*, Princeton, NJ, USA: Princeton University Press, 2003.

[16] J. Segura, “Monotonicity properties for ratios and products of modified Bessel functions and sharp trigonometric bounds”, *Results Math.*, vol. 76, art. no. 221, Oct. 2021.

[17] D. Ruiz-Antolín and J. Segura, “A new type of sharp bounds for ratios of modified Bessel functions”, *J. Math. Anal. Appl.*, vol. 443, no. 2, 15 Nov. 2016.

FRÉDÉRIC BROYDÉ was born in France in 1960. He received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d’Ingénieurs Electriciens de Grenoble (ENSIEG) and the Ph.D. in microwaves and microtechnologies from the Université des Sciences et Technologies de Lille (USTL).

He co-founded the Excem corporation in May 1988, a company providing engineering and research and development services. He is president of Excem since 1988. He is now also president and CTO of Eurexcem, a subsidiary of Excem. Most of his activity is allocated to engineering and research in electronics, radio, antennas, electromagnetic compatibility (EMC) and signal integrity.

Dr. Broydé is author or co-author of about 110 technical papers, and inventor or co-inventor of about 90 patent families, for which 78 patents of France and 49 patents of the USA have been granted. He is a Life Senior Member of the IEEE. He is a licensed radio amateur (F5OYE).

EVELYNE CLAVELIER was born in France in 1961. She received the M.S. degree in physics engineering from the Ecole Nationale Supérieure d’Ingénieurs Electriciens de Grenoble (ENSIEG).

She is co-founder of the Excem corporation, based in Maule, France, and she is currently CEO of Excem. She is also president of Tekcem, a company selling or licensing intellectual property rights to foster research. She is an active engineer and researcher.

Her current research areas are radio communications, antennas, matching networks, EMC and circuit theory.

Prior to starting Excem in 1988, she worked for Schneider Electrics (in Grenoble, France), STMicroelectronics (in Grenoble, France), and Signetics (in Mountain View, CA, USA).

Ms. Clavelier is the author or a co-author of about 100 technical papers. She is co-inventor of about 90 patent families. She is a Senior Member of the IEEE since 2002. She is a licensed radio amateur (F1PHQ).

•••

Open access license: you may reproduce, store and distribute this document, provided the document remains complete and unmodified.
All other rights reserved. Published by Excem. ISBN 978-2-909056-33-3
Excem — 12, chemin des Hauts de Clairefontaine — 78580 Maule — France

Copyright © 2026 by Excem